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Nonlinear Analysis of Two-layer Fluid Sloshing in a Rectangular Tank Subjected to Width Direction Excitation

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Abstract

This paper describes a nonlinear theory to describe oscillations of two liquid layers formed in a rectangular tank. Nonlinear equations based on the variational principle and the Galerkin method are used, which are written in a form of direct expanding in eigenmodes. Analysis using the equations is described under the experimental conditions reported in a previous study. Both the experiment and the analysis demonstrate a “peculiar oscillation” in which the interface of the two liquids oscillates at 1/5 to 1/7 of the excitation frequency in a particular excitation frequency range. By observing the nonlinear force time series, four eigenmodes are identified to be mainly relevant to the oscillation. An amplitude equation analysis was applied to the four relevant eigenmodes. It was found that this “peculiar oscillation” is one of summed and differential harmonic oscillations (combination oscillation) in which one asymmetric and one symmetric eigenmode excite each other through the mediation of other two asymmetric eigenmodes.

1 Introduction

In recent years, an increasing number of oil water separators are being installed in wastewater treatment equipment in industrial plants and soil sanitation facilities. With the increase of offshore oil production, there is also an increase in demand for oil water separators on offshore floating platforms. The major type of oil water separator consists of a separating tank in which fluids are separated by density. The two liquids form layers in the tank. It is important to predict the wave motion (sloshing) of layered fluids when considering the safety and performance of these tanks under seismic disturbances on the ground or wave disturbances on the sea.

On the subject of sloshing oscillation of two (or multi) immiscible layered fluids with different densities, Handa and Tajima [1] performed a linear analysis of two-dimensional problems in a rectangular tank and compared the results with experiments. It was reported that each linear mode (eigenmode) of sloshing contains oscillations on both of the two interfaces, the free surface interface and the interface of the two liquids, and that there are two eigenmodes having the same spatial wave number, one with

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the dominant oscillation on the free surface and the other with the dominant oscillation on the interface of the two liquids. It was also reported that, in the experiment, the free surface and the interface of the two liquids can have different mode shapes simultaneously in a tank under harmonic excitation. Tang [2] and Veletsos and Shivakumar [3] performed linear analyses for cylindrical tanks that also obtain two eigenmodes having the same spatial wave number. Xue et al. [4] conducted a sloshing experiment of two layered fluids in a rectangular tank. Molin et al. [5] conducted an experiment with a rectangular tank containing three fluids. All the previous experiments showed several patterns of interfacial oscillations on the free surface and the interface of liquids. This indicates that the sloshing actions of two (or multi) layered fluids are rather complicated in comparison with the sloshing actions of one fluid.

In sloshing oscillation of two layered fluids, nonlinear interference exists between the two interfaces. Generalis and Nagata [6] performed a perturbation based analysis in which two eigenmodes couple directly. Meziani and Ourrad [7] also performed a nonlinear analysis based on the perturbation method. La Rocca et al. [8] performed a nonlinear analysis that applied the variational principle to obtain a Lagrangian formulation and used modal damping ratios obtained by free damped vibration tests. Hara and Takahara [9] suggested a unified formulation derived by Hamiltonian mechanics and the Dirichlet-Neumann operator, which accommodates both linear and nonlinear analyses. Sciortino et al. [10] also suggested a Hamiltonian formulation. Yoshizumi [11] suggested a Lagrangian-based formulation that are written in a form of direct expanding in eigenmodes that have oscillations both on two interfaces as described above.

One of the most interesting nonlinear phenomena in previous experiments on two layered fluids sloshing is a phenomenon that has been observed by Handa and Tajima [1]. In the course of this experimental research, it was discovered that the phenomenon by which the interface of the two liquids oscillates with an asymmetric first mode shape and at a frequency of 1/5 to 1/7 of the excitation frequency. This phenomenon occurs when the excitation frequency is close to the natural frequency of the asymmetric second mode having the dominant oscillation on the free surface. This paper focuses particularly on this phenomenon, which is referred to as “peculiar oscillation.” In a previous report [11], the “peculiar oscillation” was observed by analysis using a proposed formulation. However, only frequency components in the oscillation were identified. This paper analyzes the interaction among eigenmodes in the “peculiar oscillation” to clarify the generation mechanism of the oscillation. First, three cases of the two-layer depth ratio are examined for validation. Second, based on the nonlinear modal equation, eigenmodes that mainly contribute to the “peculiar oscillation” are identified and the effects of the combination of eigenmodes on the oscillation are investigated. Finally, an amplitude equation analysis for those eigenmodes is performed. The generation mechanism of the oscillation is discussed based on the analytical results from the view point of nonlinear interactions among eigenmodes.

2 Nomenclature

a: Half the width of the tank (The width direction is along the excitation direction.)
b: Half the length of the tank in the direction horizontal and normal to the excitation direction
D_i: Domain with the i-th interface on top
F_i: i-th interface (F_0 refers to the bottom boundary.)
f: Frequency of the external disturbance (harmonic excitation) [$f = \omega/(2\pi)$]
g: Gravitational acceleration
H_i: Distance between the bottom of the tank and the i-th interface in quiescence
i: Number of the interface ($i = 1$: interface of the two liquids, $i = 2$: free surface)
j: Number of the interface at which dominant oscillation occurs in each eigenmode
($j = 1$ represents the mode in which the dominant oscillation occurs at the interface of the two

liquids. $j = 2$ represents the mode in which the dominant oscillation occurs at the free surface interface.)

N: Highest wave number order to be taken in the analysis (The same N is applied to both symmetric and asymmetric modes.)

m, n: Wave number order of eigenmodes (In this paper, wave number orders of asymmetric modes are numbered 1, 2, ... and wave number orders of symmetric modes are also numbered 1, 2, ... individually.)

p: Pressure of the liquid

t: Time

x: Coordinate in the width direction, i.e., the excitation direction (horizontal)

y: Coordinate in the direction horizontal and normal to the excitation direction

z: Coordinate in the vertical direction

(The frame, O-xyz, is attached to the tank)

x_g, X_g: Displacement of the tank excitation and its amplitude ($x_g = X_g \cdot \cos\omega t$)

ζ: Common damping ratio (In the present analyses, all ζ_{sjm} and ζ_{cjm} are set to the common value ζ.)

ζ_{sjm}, ζ_{cjm}: Critical damping ratio of the asymmetric eigenmode and that of the symmetric eigenmode

η_i: Wave height of the i-th interface

λ_{sm}, λ_{cm}: Wave number of the asymmetric eigenmode and that of the symmetric eigenmode

$$[\lambda_{sm} = (2m-1)\pi/(2a), \lambda_{cm} = 2m\pi/(2a)]$$

μ: Density ratio of the two liquids ($\mu = \rho_2/\rho_1$)

ρ_i: Density of the liquid in D_i

σ: Depth ratio ($\sigma = H_1/H_2$)

ϕ_i: Velocity potential in D_i for the liquid motion relative to the tank

ω: Angular frequency of the external disturbance (harmonic excitation) ($\omega = 2\pi f$)

ω_{sjm}, ω_{cjm}: Angular natural frequency of the asymmetric eigenmode and that of the symmetric eigenmode

3 Formulation

The formulation process of the nonlinear modal equations is described in [11]. The present paper describes the essential variational process supplemented with detailed derivation omitted in [11], and describes the resultant nonlinear modal equations used in the analysis.

3.1 Basic equation by variational principle

The analytical model is shown in Fig. 1. The basic equations are derived from the variational principle under the assumption that the two liquids are incompressible fluids and undergo an irrotational flow. The density of the Lagrangian is equal to the pressure of the liquid and the Lagrangian of the whole system, L, is then written [12] as:

$$L = \iiint_{D_1+D_2} p\,dxdydz. \tag{1}$$

The pressure p is expressed [13, 14] by the Bernoulli equation for unsteady potential flow in tanks subjected to translational motions, i.e., $p/\rho_i = -[(\partial\overline{\phi}_i/\partial t) + (1/2)(\nabla\overline{\phi}_i)^2 - \mathbf{v}_g \cdot \nabla\overline{\phi}_i + g\overline{z}]$ where, $\mathbf{v}_g$ is the excitation velocity, $\overline{z}$ is the vertical coordinate of a stationary frame, and ϕ_i is the velocity potential. In the present case, p is:

$$\frac{p}{\rho_i} = -\{\frac{\partial\phi_i}{\partial t} + \frac{1}{2}(\nabla\phi_i)^2 + gz\} - (\ddot{x}_g x - \frac{1}{2}\dot{x}_g^2), \tag{2}$$

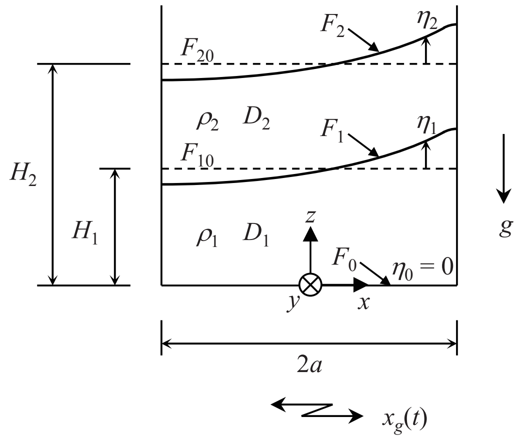


Fig. 1 Analytical model. Two incompressible immiscible fluids form layers in a rectangular tank and the tank undergoes horizontal excitation as an external disturbance.

where, the symbol $\cdot$ denotes the differentiation with respect to time, $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$, and relations of $\overline{\phi_i} = \phi_i + \dot{x}_g x$, $z = \overline{z}$, and $\mathbf{v}_g = (\dot{x}_g, 0, 0)$ are used. Substituting Eq. (2) into Eq. (1) results in:

$$\begin{aligned}\frac{L}{\rho_1} = &\iint_{F_1} (\phi_1 - \mu\phi_2)\frac{\partial \eta_1}{\partial t} dxdy - \iiint_{D_1} \{\frac{1}{2}(\nabla\phi_1)^2 + gz + \ddot{x}_g x - \frac{1}{2}\dot{x}_g^2\} dxdydz \\ &+ \mu \iint_{F_2} \phi_2 \frac{\partial \eta_2}{\partial t} dxdy - \mu \iiint_{D_2} \{\frac{1}{2}(\nabla\phi_2)^2 + gz + \ddot{x}_g x - \frac{1}{2}\dot{x}_g^2\} dxdydz \\ &- \frac{d}{dt}\iiint_{D_1} \phi_1 dxdydz - \mu \frac{d}{dt}\iiint_{D_2} \phi_2 dxdydz,\end{aligned} \tag{3}$$

where the following relationship taking into account time-dependent perturbations of F_i is used:

$$\frac{d}{dt}\iiint_{D_i} \phi_i dxdydz = \iiint_{D_i} \frac{\partial \phi_i}{\partial t} dxdydz + \iint_{F_i} \phi_i \frac{\partial \eta_i}{\partial t} dxdy - \iint_{F_{i-1}} \phi_i \frac{\partial \eta_{i-1}}{\partial t} dxdy \quad (i = 1, 2). \tag{4}$$

Hamilton's principle is given by:

$$\delta \int_{t_1}^{t_2} L dt = 0. \tag{5}$$

In the operation of the variation, η_i and ϕ_i are the respective variables. The domain D_i is the function of η_i, hence the variation in D_i is written as:

$$\delta \iiint_{D_i} \mathscr{F}(t,x,y,z) dxdydz = \iiint_{D_i} \delta \mathscr{F} dxdydz + \iint_{F_i} \mathscr{F} \delta\eta_i dxdy - \iint_{F_{i-1}} \mathscr{F} \delta\eta_{i-1} dxdy, \ (i = 1, 2), \tag{6}$$

where, $\mathscr{F}$ represents a general function, and $\delta\eta_0 = 0$.

Applying Eqs. (6) to Eq. (5) with Eq. (3) under constraints $\delta\phi_i = 0$ at $t = t_1$ and $t = t_2$ obtains:

$$\begin{aligned}&\int_{t_1}^{t_2} \iint_{F_1} (\delta\phi_1 - \mu\delta\phi_2)\frac{\partial \eta_1}{\partial t} dxdydt + \int_{t_1}^{t_2} \iint_{F_1} (\phi_1 - \mu\phi_2)\frac{\partial \delta\eta_1}{\partial t} dxdydt + \mu \int_{t_1}^{t_2} \iint_{F_2} \delta\phi_2 \frac{\partial \eta_2}{\partial t} dxdydt \\ &+ \mu \int_{t_1}^{t_2} \iint_{F_2} \phi_2 \frac{\partial \delta\eta_2}{\partial t} dxdydt + \int_{t_1}^{t_2} [-\iint_{F_1} \{\frac{1}{2}(\nabla\phi_1)^2 + gz + \ddot{x}_g x - \frac{1}{2}\dot{x}_g^2\}\delta\eta_1 dxdy \\ &- \iiint_{D_1} \frac{1}{2}\delta(\nabla\phi_1)^2 dxdydz + \mu \iint_{F_1} \{\frac{1}{2}(\nabla\phi_2)^2 + gz + \ddot{x}_g x - \frac{1}{2}\dot{x}_g^2\}\delta\eta_1 dxdy \\ &- \mu \iint_{F_2} \{\frac{1}{2}(\nabla\phi_2)^2 + gz + \ddot{x}_g x - \frac{1}{2}\dot{x}_g^2\}\delta\eta_2 dxdy - \mu \iiint_{D_2} \frac{1}{2}\delta(\nabla\phi_2)^2 dxdydz] dt = 0.\end{aligned} \tag{7}$$

With respect to the second and fourth terms in Eq. (7), applying integration by parts to integrating $\phi_i(\partial\delta\eta_i/\partial t)$ under constraints $\delta\eta_i = 0$ at $t = t_1$ and $t = t_2$ yields:

$$\int_{t_1}^{t_2}\iint_{F_1}(\phi_1-\mu\phi_2)\frac{\partial\delta\eta_1}{\partial t}dxdydt = -\int_{t_1}^{t_2}\iint_{F_1}(\frac{\partial\phi_1}{\partial t}-\mu\frac{\partial\phi_2}{\partial t})\delta\eta_1 dxdydt, \tag{8}$$

$$\int_{t_1}^{t_2}\iint_{F_2}\phi_2\frac{\partial\delta\eta_2}{\partial t}dxdydt = -\int_{t_1}^{t_2}\iint_{F_2}\frac{\partial\phi_2}{\partial t}\delta\eta_2 dxdydt. \tag{9}$$

Here Green's first identity is recalled, i.e., $\iiint_D \nabla\phi\,\nabla\psi dxdydz = \iint_{\partial D}\phi\cdot(\partial\psi/\partial n)ds - \iiint_D \phi\,\nabla^2\psi dxdydz$ where D is the domain, ∂D is its closed boundary, n is the outward pointing normal of the boundary ∂D, ds is the surface element of ∂D, and ϕ and ψ are functions. Applying Green's first identity to integrating $\delta(\nabla\phi_i)^2$in Eq. (7) and expanding variations in the form of $\delta(\alpha\beta) = (\delta\alpha)\beta + \alpha(\delta\beta)$ yields:

$$\begin{aligned}\iiint_{D_1}\delta(\nabla\phi_1)^2 dxdydz = &\iint_{F_1}(\delta\phi_1\frac{\partial\phi_1}{\partial n}+\phi_1\frac{\partial\delta\phi_1}{\partial n})ds + \iint_{\partial D_1-F_1}(\delta\phi_1\frac{\partial\phi_1}{\partial n}+\phi_1\frac{\partial\delta\phi_1}{\partial n})ds \\ &-\iiint_{D_1}(\delta\phi_1\nabla^2\phi_1+\phi_1\nabla^2\delta\phi_1)dxdydz,\end{aligned} \tag{10}$$

$$\begin{aligned}\iiint_{D_2}\delta(\nabla\phi_2)^2 dxdydz = &-\iint_{F_1}(\delta\phi_2\frac{\partial\phi_2}{\partial n}+\phi_2\frac{\partial\delta\phi_2}{\partial n})ds + \iint_{F_2}(\delta\phi_2\frac{\partial\phi_2}{\partial n}+\phi_2\frac{\partial\delta\phi_2}{\partial n})ds \\ &+\iint_{\partial D_2-F_1-F_2}(\delta\phi_2\frac{\partial\phi_2}{\partial n}+\phi_2\frac{\partial\delta\phi_2}{\partial n})ds - \iiint_{D_2}(\delta\phi_2\nabla^2\phi_2+\phi_2\nabla^2\delta\phi_2)dxdydz,\end{aligned} \tag{11}$$

where, ∂D_i is the closed boundary of the domain D_i, ds is the surface element of ∂D_i and n is the outward pointing normal of the boundary ∂D_i except on F_1 of ∂D_2. The positive direction of n on F_1 is defined to be the $+z$ side, and n is the inward pointing normal on F_1 of ∂D_2. The last terms in Eqs. (10) and (11) are rewritten as follows by using Green's first identity again:

$$\begin{aligned}\iiint_{D_1}\phi_1\nabla^2\delta\phi_1 dxdydz = &\iint_{F_1}(-\delta\phi_1\frac{\partial\phi_1}{\partial n}+\phi_1\frac{\partial\delta\phi_1}{\partial n})ds + \iint_{\partial D_1-F_1}(-\delta\phi_1\frac{\partial\phi_1}{\partial n}+\phi_1\frac{\partial\delta\phi_1}{\partial n})ds \\ &+\iiint_{D_1}\delta\phi_1\nabla^2\phi_1 dxdydz,\end{aligned} \tag{12}$$

$$\begin{aligned}\iiint_{D_2}\phi_2\nabla^2\delta\phi_2 dxdydz = &-\iint_{F_1}(-\delta\phi_2\frac{\partial\phi_2}{\partial n}+\phi_2\frac{\partial\delta\phi_2}{\partial n})ds + \iint_{F_2}(-\delta\phi_2\frac{\partial\phi_2}{\partial n}+\phi_2\frac{\partial\delta\phi_2}{\partial n})ds \\ &+\iint_{\partial D_2-F_1-F_2}(-\delta\phi_2\frac{\partial\phi_2}{\partial n}+\phi_2\frac{\partial\delta\phi_2}{\partial n})ds + \iiint_{D_2}\delta\phi_2\nabla^2\phi_2 dxdydz.\end{aligned} \tag{13}$$

The following relationship for the interfaces is used in the differentiation with respect to the normal of the interface:

$$(\frac{\partial\phi_i}{\partial n})ds = (\frac{\partial\phi_i}{\partial z}-\frac{\partial\phi_i}{\partial x}\frac{\partial\eta}{\partial x}-\frac{\partial\phi_i}{\partial y}\frac{\partial\eta}{\partial y})dxdy, \tag{14}$$

where, η is η_1 for $F = F_1$ and η_2 for $F = F_2$. After substituting Eqs. (12) and (13) into $\iiint_{D_i}\phi_i\nabla^2\delta\phi_i dxdydz$

of Eqs. (10) and (11), substituting Eqs. (8) to (11) into Eq. (7) and using Eq. (14) obtains:

$$\begin{aligned}
&\int_{t_1}^{t_2}[\iint_{F_1}\{(\frac{\partial\eta_1}{\partial t}-\frac{\partial\phi_1}{\partial z}+\frac{\partial\phi_1}{\partial x}\frac{\partial\eta_1}{\partial x}+\frac{\partial\phi_1}{\partial y}\frac{\partial\eta_1}{\partial y})\delta\phi_1-\mu(\frac{\partial\eta_1}{\partial t}-\frac{\partial\phi_2}{\partial z}+\frac{\partial\phi_2}{\partial x}\frac{\partial\eta_1}{\partial x}+\frac{\partial\phi_2}{\partial y}\frac{\partial\eta_1}{\partial y})\delta\phi_2\}dxdy\\
&+\mu\iint_{F_2}(\frac{\partial\eta_2}{\partial t}-\frac{\partial\phi_2}{\partial z}+\frac{\partial\phi_2}{\partial x}\frac{\partial\eta_2}{\partial x}+\frac{\partial\phi_2}{\partial y}\frac{\partial\eta_2}{\partial y})\delta\phi_2 dxdy\\
&-\iint_{F_1}[\frac{\partial\phi_1}{\partial t}+\frac{1}{2}(\nabla\phi_1)^2+gz+\ddot{x}_g x-\frac{1}{2}\dot{x}_g^2-\mu\{\frac{\partial\phi_2}{\partial t}+\frac{1}{2}(\nabla\phi_2)^2+gz+\ddot{x}_g x-\frac{1}{2}\dot{x}_g^2\}]\delta\eta_1 dxdy\\
&-\mu\iint_{F_2}\{\frac{\partial\phi_2}{\partial t}+\frac{1}{2}(\nabla\phi_2)^2+gz+\ddot{x}_g x-\frac{1}{2}\dot{x}_g^2\}\delta\eta_2 dxdy-\iint_{\partial D_1-F_1}\frac{\partial\phi_1}{\partial n}\delta\phi_1 ds\\
&-\mu\iint_{\partial D_2-F_1-F_2}\frac{\partial\phi_2}{\partial n}\delta\phi_2 ds+\iiint_{D_1}\nabla^2\phi_1\delta\phi_1 dxdydz+\mu\iiint_{D_2}\nabla^2\phi_2\delta\phi_2 dxdydz]dt=0.
\end{aligned}\tag{15}$$

By assuming the following Eqs. (18) to (19), the kinematic conditions and dynamic conditions imposed on the interfaces are separated from Eq. (15) to obtain Eqs. (16) and (17) respectively.

$$\begin{aligned}
&\iint_{F_1}\{(\frac{\partial\eta_1}{\partial t}-\frac{\partial\phi_1}{\partial z}+\frac{\partial\phi_1}{\partial x}\frac{\partial\eta_1}{\partial x}+\frac{\partial\phi_1}{\partial y}\frac{\partial\eta_1}{\partial y})\delta\phi_1-\mu(\frac{\partial\eta_1}{\partial t}-\frac{\partial\phi_2}{\partial z}+\frac{\partial\phi_2}{\partial x}\frac{\partial\eta_1}{\partial x}+\frac{\partial\phi_2}{\partial y}\frac{\partial\eta_1}{\partial y})\delta\phi_2\}dxdy\\
&\qquad+\mu\iint_{F_2}(\frac{\partial\eta_2}{\partial t}-\frac{\partial\phi_2}{\partial z}+\frac{\partial\phi_2}{\partial x}\frac{\partial\eta_2}{\partial x}+\frac{\partial\phi_2}{\partial y}\frac{\partial\eta_2}{\partial y})\delta\phi_2 dxdy=0,
\end{aligned}\tag{16}$$

$$\begin{aligned}
&\iint_{F_1}[\frac{\partial\phi_1}{\partial t}-\mu(\frac{\partial\phi_2}{\partial t})+\frac{1}{2}\{(\nabla\phi_1)^2-\mu(\nabla\phi_2)^2\}+(1-\mu)gz+(1-\mu)(\ddot{x}_g x-\frac{1}{2}\dot{x}_g^2)]\delta\eta_1 dxdy\\
&\qquad+\mu\iint_{F_2}\{\frac{\partial\phi_2}{\partial t}+\frac{1}{2}(\nabla\phi_2)^2+gz+\ddot{x}_g x-\frac{1}{2}\dot{x}_g^2\}\delta\eta_2 dxdy=0,
\end{aligned}\tag{17}$$

$$\frac{\partial\phi_1}{\partial n}=0 \text{ on } \partial D_1-F_1, \frac{\partial\phi_2}{\partial n}=0 \text{ on } \partial D_2-F_1-F_2, \tag{18}$$

$$\nabla^2\phi_1=0 \text{ in } D_1, \nabla^2\phi_2=0 \text{ in } D_2 \tag{19}$$

Equations (16) and (17) are written in integral forms combining two interface conditions. These integral forms are convenient to formulate the nonlinear coupling between two interfaces through the direct variational method, e.g., the Galerkin method in the following section. Equation (18) gives the boundary conditions imposed on the wall of the tank. Equation (19) is Laplace's equation that corresponds to the incompressibility condition.

3.2 Nonlinear modal equations

In the remainder of this analysis, the problem is restricted to a two-dimensional flow in a rectangular tank. Length is reduced by a. Time is reduced by the natural frequency of the eigenmode under the linear resonance state, ω_{res}. The symbol '*', which means dimensionless, is omitted at equations in the remainder.

$$\left.\begin{aligned}
&a=aa^*,\quad g=a\omega_{res}^2 g^*,\quad H_i=aH_i^*,\quad t=t^*/\omega_{res},\quad x=ax^*,\quad z=az^*,\\
&x_g=ax_g^*,\quad X_g=aX_g^*,\quad \eta_i=a\eta_i^*,\quad \lambda_{sm}=\lambda_{sm}^*/a,\quad \lambda_{cm}=\lambda_{cm}^*/a,\quad \phi_i=a^2\omega_{res}\phi_i^*,\\
&\omega_{sjm}=\omega_{res}\omega_{sjm}^*,\quad \omega_{cjm}=\omega_{res}\omega_{cjm}^*
\end{aligned}\right\}\tag{20}$$

An approximate solution of the basic equations was sought, applying the Galerkin method. The admissible functions are expressed by direct eigenmode expansion. The procedure for obtaining eigenmodes is shown in Appendix 1.

$$\phi_i(x,z,t) = \sum_{j=1}^{2}\sum_{m}(\underbrace{\Phi_{sijm}Z_{sijm}(z)\sin(\lambda_{sm}x)A_{jm}(t)}_{\phi_{sijm}(x,t)} + \underbrace{\Phi_{cijm}Z_{cijm}(z)\cos(\lambda_{cm}x)B_{jm}(t)}_{\phi_{cijm}(x,t)}), \tag{21}$$

$$\eta_i(x,t) = \sum_{j=1}^{2}\sum_{m}(\underbrace{\Phi_{sijm}Z'_{sijm}(H_i)\sin(\lambda_{sm}x)C_{jm}(t)}_{\eta_{sijm}(x,t)} + \underbrace{\Phi_{cijm}Z'_{cijm}(H_i)\cos(\lambda_{cm}x)D_{jm}(t)}_{\eta_{cijm}(x,t)}), \tag{22}$$

where, ϕ_{sijm} and ϕ_{cijm} are velocity potentials of asymmetric and symmetric eigenmodes respectively, Z_{sijm} and Z_{cijm} are mode shapes in the z-direction of the velocity potentials in the eigenmodes, the prime denotes d/dz. A_{jm} and C_{jm} are time dependent variables for the asymmetric eigenmodes. B_{jm} and D_{jm} are time dependent variables for the symmetric eigenmodes. The admissible function of Eq. (21) satisfies the boundary conditions of Eq. (18). Orders of quantities are assumed as follows:

$$A_{jm}, B_{jm}, C_{jm}, D_{jm} \sim O(\varepsilon^{1.5}), \tag{23}$$

where, $O(\varepsilon)$ represents the ε-order, and $\varepsilon = X_g^{*1/3}$. In resonance cases in which one eigenmode is dominant, the order of the eigenmode is $O(\varepsilon)$ [11]. In the "peculiar oscillation" discussed in the present paper, a few eigenmodes are dominant simultaneously and no eigenmode is assumed to be $O(\varepsilon)$.

After applying the Galerkin method, the resultant nonlinear modal equations are obtained as follows:

$$\begin{aligned}
&\Gamma_{1sjn}\ddot{C}_{jn} + 2\zeta_{sjn}\omega_{sjn}\Gamma_{1sjn}\dot{C}_{jn} + \Gamma_{2sjn}C_{jn} - (-1)^n\left\{2/(\lambda_{sn}^2 a)\right\}P_{gjn}\ddot{x}_g \\
&+\frac{1}{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{m=1}^{N-n+1}(P_{1sj_1j_2j,m,m+n-1,n}\ddot{C}_{j_1m}D_{j_2m+n-1} + P_{2sj_1j_2j,m,m+n-1,n}\dot{C}_{j_1m}\dot{D}_{j_2m+n-1}) \\
&-\frac{1}{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{m=n+1}^{N}(P_{1sj_1j_2j,m,m-n,n}\ddot{C}_{j_1m}D_{j_2m-n} + P_{2sj_1j_2j,m,m-n,n}\dot{C}_{j_1m}\dot{D}_{j_2m-n}) \\
&-\frac{1}{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{m=1}^{n-1}(P_{3sj_1j_2j,m,n-m,n}\ddot{C}_{j_1m}D_{j_2n-m} + P_{4sj_1j_2j,m,n-m,n}\dot{C}_{j_1m}\dot{D}_{j_2n-m}) \\
&-\frac{1}{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{m=1}^{N-n}(P_{5sj_1j_2j,m,m+n,n}\ddot{D}_{j_1m}C_{j_2m+n} + P_{6sj_1j_2j,m,m+n,n}\dot{D}_{j_1m}\dot{C}_{j_2m+n}) \\
&+\frac{1}{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{m=n}^{N}(P_{5sj_1j_2j,m,m-n+1,n}\ddot{D}_{j_1m}C_{j_2m-n+1} + P_{6sj_1j_2j,m,m-n+1,n}\dot{D}_{j_1m}\dot{C}_{j_2m-n+1}) \\
&-\frac{1}{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{m=1}^{n-1}(P_{7sj_1j_2j,m,n-m,n}\ddot{D}_{j_1m}C_{j_2n-m} + P_{8sj_1j_2j,m,n-m,n}\dot{D}_{j_1m}\dot{C}_{j_2n-m}) \\
=&0 \quad (j=1,2, n=1,2,\ldots,N),
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\Gamma_{1cjn}\ddot{D}_{jn} + 2\zeta_{cjn}\omega_{cjn}\Gamma_{1cjn}\dot{D}_{jn} + \Gamma_{2cjn}D_{jn} \\
&+\frac{1}{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{m=1}^{n}(P_{1cj_1j_2j,m,n-m+1,n}\ddot{C}_{j_1m}C_{j_2n-m+1} + P_{2cj_1j_2j,m,n-m+1,n}\dot{C}_{j_1m}\dot{C}_{j_2n-m+1}) \\
&-\frac{1}{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{m=n+1}^{N}(P_{3cj_1j_2j,m,m-n,n}\ddot{C}_{j_1m}C_{j_2m-n} + P_{4cj_1j_2j,m,m-n,n}\dot{C}_{j_1m}\dot{C}_{j_2m-n})
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{m=1}^{N-n}(P_{3cj_1j_2j,m,m+n,n}\ddot{C}_{j_1m}C_{j_2m+n}+P_{4cj_1j_2j,m,m+n,n}\dot{C}_{j_1m}\dot{C}_{j_2m+n})\\
&-\frac{1}{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{m=1}^{n-1}(P_{5cj_1j_2j,m,n-m,n}\ddot{D}_{j_1m}D_{j_2n-m}+P_{6cj_1j_2j,m,n-m,n}\dot{D}_{j_1m}\dot{D}_{j_2n-m})\\
&-\frac{1}{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{m=n+1}^{N}(P_{7cj_1j_2j,m,m-n,n}\ddot{D}_{j_1m}D_{j_2m-n}+P_{8cj_1j_2j,m,m-n,n}\dot{D}_{j_1m}\dot{D}_{j_2m-n})\\
&-\frac{1}{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{m=1}^{N-n}(P_{7cj_1j_2j,m,m+n,n}\ddot{D}_{j_1m}D_{j_2m+n}+P_{8cj_1j_2j,m,m+n,n}\dot{D}_{j_1m}\dot{D}_{j_2m+n})\\
=&0 \quad (j=1,2,n=1,2,\ldots,N). \qquad (25)
\end{aligned}
$$

Viscous damping terms are introduced as the second terms in order to express the equivalent energy dissipation. The process of the Galerkin method to obtain Eqs. (24) and (25) is summarized in Appendix 2. The specific expressions of the symbols (Γ_{1sjn}, Γ_{2sjn}, Γ_{1cjn}, Γ_{2cjn}, P_{gjn}, $P_{1-8}\ldots$) in Eqs. (24) and (25) are given in [11].

In previous research, only Generalis and Nagata [6] showed an analysis method treating the direct coupling of eigenmodes (not wave numbers). However, this research examined only two eigenmodes. The nonlinear equations of Eqs (24) and (25) are written in a form of direct coupling of several eigenmodes. This allows the direct analysis of nonlinear coupling among eigenmodes.

The equation set of Eqs. (24) and (25) can be expressed in a form of state-space representation as $[\boldsymbol{A}]\dot{q}=\boldsymbol{B}(\boldsymbol{q})$ where $[\boldsymbol{A}]$ is the state matrix, $\boldsymbol{B}(\boldsymbol{q})$ is the input vector and $\boldsymbol{q}$ is the state vector. However, since nonlinear terms include the highest derivative with respect to time (the second derivative in this case), the state matrix $[\boldsymbol{A}]$ is not a constant matrix but a function matrix of $\boldsymbol{q}$. Then, the state-space representation being solved is obtained as $\dot{q}=[\boldsymbol{A}(\boldsymbol{q})]^{-1}\boldsymbol{B}(\boldsymbol{q})$. In the time domain analysis, the equation set was solved using the Runge-Kutta method (RK4). The time division in the numerical simulation was set to 1/100 of the harmonic excitation period. The analysis time length was set to 100 periods basically to obtain a stationary state, and was extended to obtain a stationary state if 100 periods was not sufficient. Each analysis was performed under both conditions of increasing and decreasing excitation frequency. The wave heights obtained by Eqs. (24) and (25) are consistent well with those in the experiment over the ω-region including several natural frequencies [11].

Linear solutions can be obtained by omitting the nonlinear terms multiplied by '$P_{1-8}\ldots$' in Eqs. (24) and (25). The linear solution has been confirmed to coincide with that by Hara et al. [9].

4 Time domain analysis

The time domain analysis using Eqs. (24) and (25) was performed under the experimental conditions reported by Handa and Tajima [1]. Only one depth ratio of the two layers was examined in the previous report [11]. In contrast, the present paper examines three experimental cases.

4.1 Analytical conditions and natural frequencies

The conditions are listed in Table 1. In the experiment, water and kerosene were used and the density ratio of the fluids was 0.789 [1]. The natural frequencies and mode shapes obtained by the eigenmode analysis (Appendix 1) are shown in Table 2. In the remainder of this paper, an eigenmode is described as a number row that consists of the number of the interface with the dominant oscillation (j) and the wave number order of eigenmodes (n), for example, the asymmetric first mode ($n = 1$) with the dominant oscillation on the free surface interface F_2 ($j = 2$) is called the asymmetric (2, 1) mode.

Table 1 Analytical conditions (The conditions correspond to those in the experiments performed by Handa and Tajima [1]).

Condition	(a)	(b)	(c)
Length of tank $2a$ [m]		0.400	
Width of tank $2b$ [m]		0.100	
Depth H_1 [m]	0.100	0.050	0.150
Depth H_2 [m]		0.200	
$\sigma = H_1/H_2$	0.50	0.25	0.75
Ratio of density μ		0.789	
Acceleration of gravity g [m/s^2]		9.8	
Excitation amplitude X_g [m]		0.010	
Damping ratio ζ⋇		0.020, 0.050	

⋇ The common damping ratio is applied to all modes.

In the experiment, the tank was exposed to harmonic excitation as an external disturbance, and x_g is set to $x_g = X_g \cdot \cos \omega t$ in the analysis. Eight eigenmodes were incorporated in the analysis, i.e., eigenmodes of the first or the second order wave number of asymmetric or symmetric mode shapes with the dominant oscillation on the interface of two liquids or on the free surface interface. The same critical modal damping ratio (assumed to be 2% and 5%) was applied to all eight eigenmodes.

4.2 Interface oscillation under each excitation frequency

The experiment identified "peculiar oscillation," i.e. when the asymmetric (1, 1) mode oscillates at a frequency of 1/5 to 1/7 of the excitation frequency, which occurs at the excitation frequency range of 2.1 Hz to 2.3 Hz around the resonance frequency of the asymmetric (2, 2) mode [1]. This phenomenon was also observed in the present nonlinear analysis. In the three depth ratio cases, the maximum wave heights observed in the stationary state under each excitation frequency are plotted in Fig. 2. The maximum wave height is the maximum positive ($+z$ side) change measured from a quiescent state. Transitions of the interface profiles under the "peculiar oscillation" are illustrated for the three depth ratio cases in Fig. 3.

The nonlinear analysis demonstrates the "peculiar oscillation" in all three depth ratio cases in the same way as the experiment as shown in Fig. 2. Linear solutions are also shown in Fig. 2 and do not demonstrate "peculiar oscillation". In the analysis, the excitation frequency range of the "peculiar oscillation" in which η_1 develops becomes wider with the increasing depth ratio σ. However, the change in the experimental frequency range is unclear. In the larger σ the two interfaces become close and it may be considered that nonlinear interference is occurring and leading to a wider "peculiar oscillation" frequency range.

Figure 3 shows profile transitions in three excitation periods. Here, it can be observed that the two-liquid interface (η_1) makes an oscillation of about 1/2 period with the mode shape of the asymmetric (1, 1) mode during three periods of harmonic excitation. The two-liquid interface (η_1) at $\sigma = 0.75$ has other mode shapes and the asymmetric (1, 1) mode oscillation is somewhat unclear in the profile transition. At $\sigma = 0.75$, the two-liquid interfaces are close and eigenmodes with the dominant oscillation at η_2 (eigenmodes of $j = 2$) have significant oscillations at η_1 [see Table 2 (c)]. Then, the mode shapes of the $j = 2$ eigenmodes are superimposed at η_1 under the "peculiar oscillation". In contrast, the free surface interface (η_2) makes an oscillation of one period during one period of excitation regardless of the depth ratio. In all three depth ratio cases, η_2 seems to have a few mode shapes including the asymmetric (2, 2) mode.

Table 2 Natural frequencies and mode shapes by eigenmode analysis.

Condition (a) ($\sigma = 0.50$)

Dominant interface	1st asymmetric	1st symmetric	2nd asymmetric	2nd symmetric
F_1 ($j = 1$)	asym. (1, 1) 0.382 Hz	sym. (1, 1) 0.646 Hz	asym. (1, 2) 0.822 Hz	sym. (1, 2) 0.957 Hz
F_2 ($j = 2$)	asym. (2, 1) 1.328 Hz	sym. (2, 1) 1.970 Hz	asym. (2, 2) 2.418 Hz	sym. (2, 2) 2.793 Hz

Condition (b) ($\sigma = 0.25$)

Dominant interface	1st asymmetric	1st symmetric	2nd asymmetric	2nd symmetric
F_1 ($j = 1$)	asym. (1, 1) 0.336 Hz	sym. (1, 1) 0.594 Hz	asym. (1, 2) 0.786 Hz	sym. (1, 2) 0.936 Hz
F_2 ($j = 2$)	asym. (2, 1) 1.330 Hz	sym. (2, 1) 1.970 Hz	asym. (2, 2) 2.418 Hz	sym. (2, 2) 2.793 Hz

Condition (c) ($\sigma = 0.75$)

Dominant interface	1st asymmetric	1st symmetric	2nd asymmetric	2nd symmetric
F_1 ($j = 1$)	asym. (1, 1) 0.336 Hz	sym. (1, 1) 0.594 Hz	asym. (1, 2) 0.786 Hz	sym. (1, 2) 0.936 Hz
F_2 ($j = 2$)	asym. (2, 1) 1.330 Hz	sym. (2, 1) 1.970 Hz	asym. (2, 2) 2.418 Hz	sym. (2, 2) 2.793 Hz

5 Eigenmodes involved in "peculiar oscillation"

In the "peculiar oscillation," the two-liquid interface, η_1, oscillates with the asymmetric (1, 1) mode at a frequency of 1/5 to 1/7 of the excitation frequency. The mechanism of this oscillation is discussed below.

5.1 Causal nonlinear forces and frequency components under "peculiar oscillation"

By observing the time series of all nonlinear terms in Eq. (24) acting on the asymmetric (1, 1) mode $[C_{11}]$ under the "peculiar oscillation," some terms have been identified as nonlinear force terms having a component clearly synchronized to the asymmetric (1, 1) mode oscillation and that then excite the asymmetric (1, 1) mode. These terms are classified into two groups. One is the group of four terms expressed as products of the asymmetric (2, 1) mode $[C_{21}]$ and the symmetric (2, 1) mode $[D_{21}]$, and

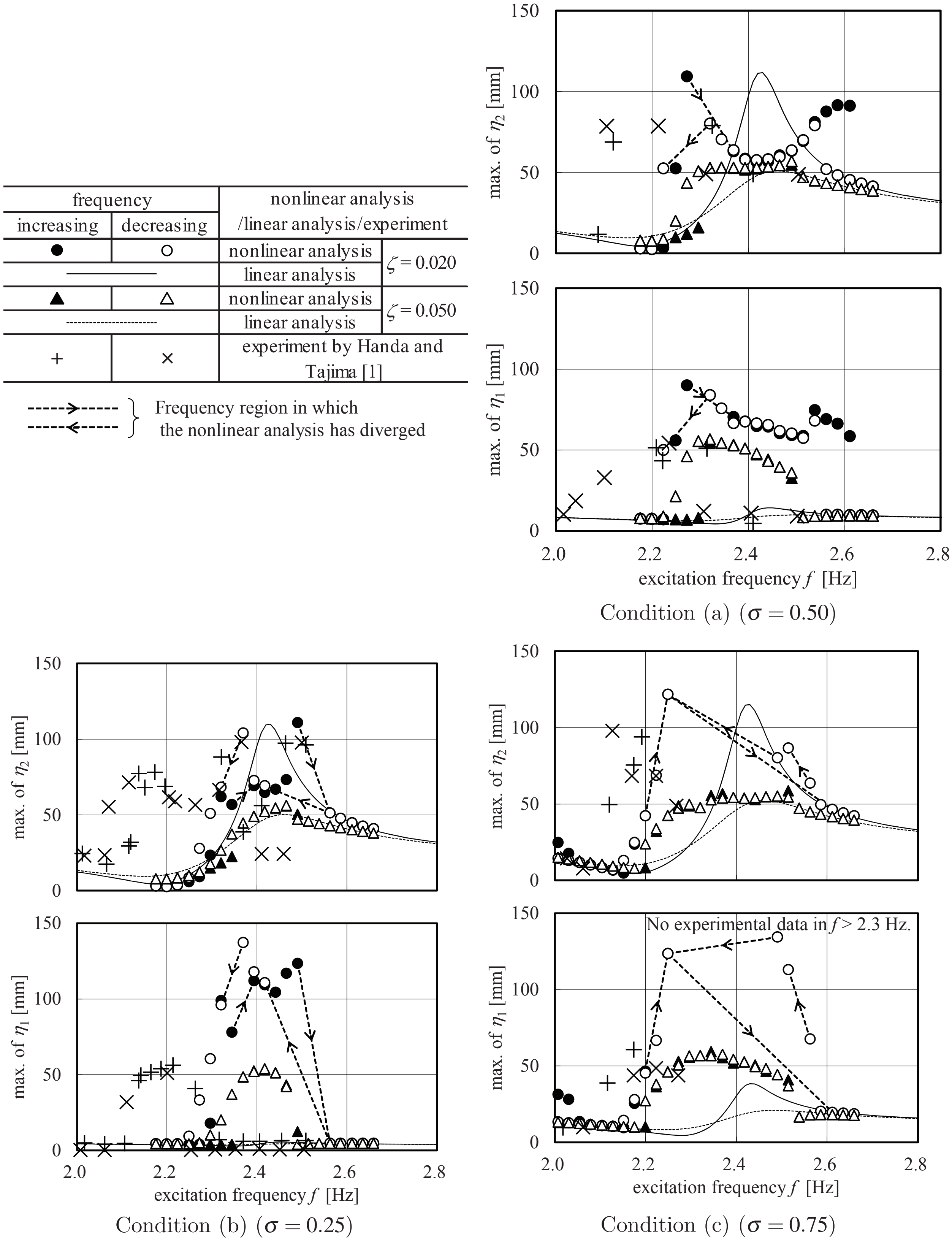


Fig. 2 Maximum response of interfaces ($x = a$).

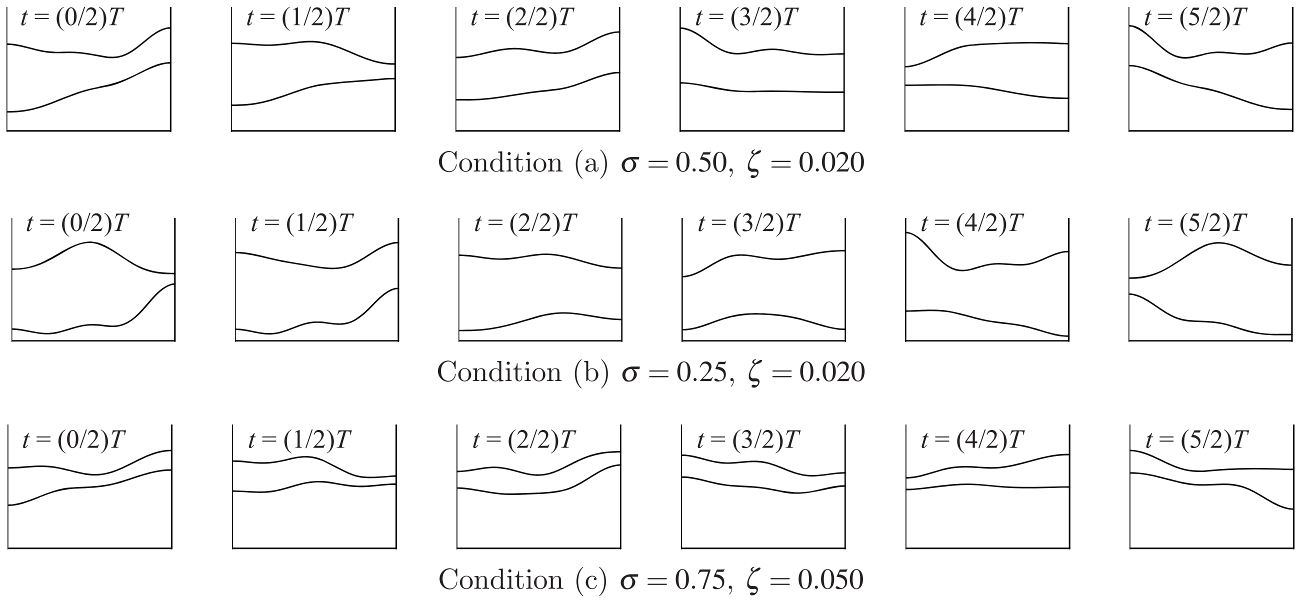


Fig. 3 Wave height profile under "peculiar oscillation" [analysis, excitation frequency $f = 2.418$ Hz (period: $T = 1/f = 0.414$ sec.)

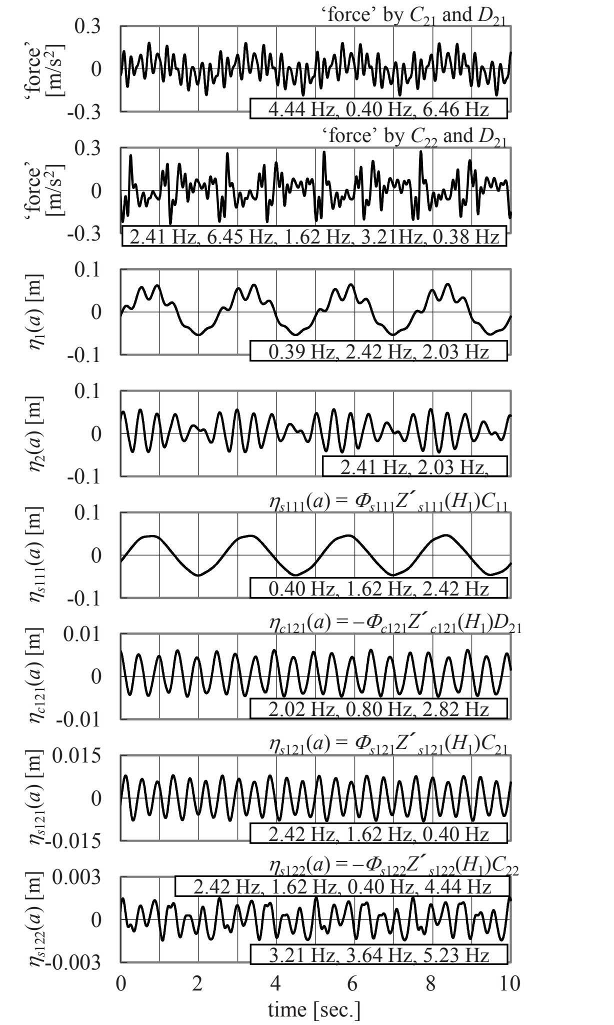
the other is the group of four terms expressed as products of the asymmetric (2, 2) mode [C_{22}] and the symmetric (2, 1) mode [D_{21}]. The time series of the sum of four terms consisting of C_{21} and D_{21} and that of four terms consisting of C_{22} and D_{21} are shown in the upper two time series in Fig. 4 ('force').

The wave heights on the interface of the two liquids and on the free surface interface, $\eta_1(a)$ and $\eta_2(a)$, are also shown in Fig. 4. The components of the relevant four eigenmodes are separated from $\eta_1(a)$ and are also shown in Fig. 4, i.e., $\eta_{s111}(a)$, $\eta_{c121}(a)$, $\eta_{s121}(a)$ and $\eta_{s122}(a)$. The frequencies written on the time series in Fig. 4 are the dominant frequencies in FFT analysis. The power spectra of the relevant four eigenmodes by FFT are shown in Fig. 5.

The largest frequency component in the asymmetric (1, 1) mode [C_{11}] shown in Fig. 5 is 0.40 Hz [$\omega_1/(2\pi)$]. This is close to the natural frequency of the mode itself (0.382 Hz), Then, the asymmetric (1, 1) mode [C_{11}] oscillates mainly at a frequency close to its natural frequency. From the 'force' in Fig. 4, it can be seen that the nonlinear terms have a frequency component of 0.40 Hz or 0.38 Hz. Thus, the asymmetric (1, 1) mode [C_{11}] is excited by nonlinear terms having a frequency component close to the mode's natural frequency, and the asymmetric (1, 1) mode [C_{11}] is under a nonlinear resonance.

With respect to other three eigenmodes, the spectra in Fig. 5 indicate that the symmetric (2, 1) mode [D_{21}] also has the largest component [2.02 Hz, $\omega_2/(2\pi)$] close to its natural frequency (1.970 Hz), while both the asymmetric (2, 1) mode [C_{21}] and the asymmetric (2, 2) mode [C_{22}] have the largest component of the excitation frequency [2.42 Hz, $\omega/\ (2\pi)$]. Consequently, two eigenmodes (C_{11} and D_{21}) oscillate mainly at a frequency close to the natural frequency, respectively, and are asynchronous to the excitation, and the other two eigenmodes (C_{21} and C_{22}) oscillate mainly at the excitation frequency. There is a relationship of $\omega_1 + \omega_2 = \omega$ among the three frequencies. This is the frequency relationship characterizing summed and differential harmonic oscillations (combination oscillations) [15].

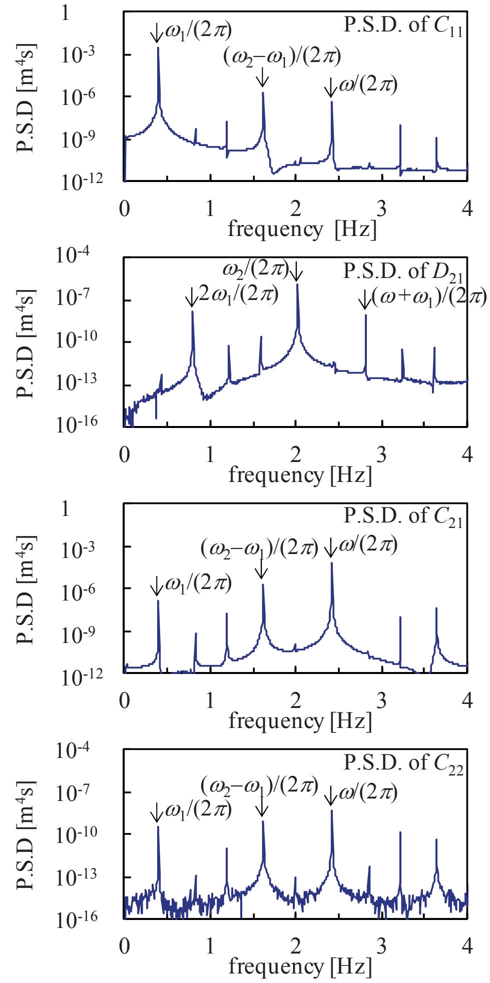
With respect to the two sums of nonlinear terms ('force' in Fig. 4, i.e., the sum consisting of C_{21} and D_{21} and that consisting of C_{22} and D_{21}), shown in Fig. 6 are asymmetric (1, 1) mode-synchronous components under each excitation frequency. These synchronous components are obtained by applying the 4th order Butterworth filter (band-pass filter). The filtering band is $0.5 \times fs_{11}$ to $1.5 \times fs_{11}$ where fs_{11} is the natural frequency of the asymmetric (1, 1) mode (close to the "peculiar oscillation" frequency). As shown in Fig. 6, although the magnitude relationship between the two sums depends



$$\text{'force' by } C_{21} \text{ and } D_{21} = (1/2)\left(P_{1s221111}\ddot{C}_{21}D_{21} + P_{2s221111}\dot{C}_{21}\dot{D}_{21} + P_{5s221111}\ddot{D}_{21}C_{21} + P_{6s221111}\dot{D}_{21}\dot{C}_{21}\right)$$

$$\text{'force' by } C_{22} \text{ and } D_{21} = -(1/2)\left(P_{1s221211}\ddot{C}_{22}D_{21} + P_{2s221211}\dot{C}_{22}\dot{D}_{21} + P_{5s221121}\ddot{D}_{21}C_{22} + P_{6s221121}\dot{D}_{21}\dot{C}_{22}\right)$$

Fig. 4 Time series of "peculiar oscillation" (analysis, σ = 0.50, ζ = 0.02 f = 2.418 Hz).



ω_1: frequency close to the natural frequency of the asymmetric (1, 1) mode
ω_2: frequency close to the natural frequency of the symmetric (2, 1) mode
ω: excitation (external disturbance) frequency

Fig. 5 Power spectral densities of the relevant four modes, i.e. asym. (1, 1) (C_{11}), sym. (2, 1) (D_{21}), asym. (2, 1) (C_{21}) and asym. (2, 2) (C_{22}). (analysis, $\sigma = 0.50$, $\zeta = 0.02$, $f = 2.418$ Hz).

on the excitation frequency, both sums have the same order magnitude. This means that both sums of the nonlinear terms act on the asymmetric (1, 1) mode and contribute to the "peculiar oscillation" in parallel.

5.2 Effect of combination of eigenmodes on "peculiar oscillation"

The "peculiar oscillation" can be regarded as a nonlinear resonance phenomenon consisting of two eigenmodes under the nonlinear resonance (the asymmetric (1, 1) mode [C_{11}], the symmetric (2, 1) mode [D_{21}]) and other eigenmodes excited by the external disturbance. The effects of eigenmodes other than C_{11} and D_{21} are examined below.

Three cases of eigenmode combinations were analyzed as well as the original analysis using all eight modes. The cases are listed in the top of Fig. 7. One case is 4-mode analysis using two eigenmodes (C_{11}

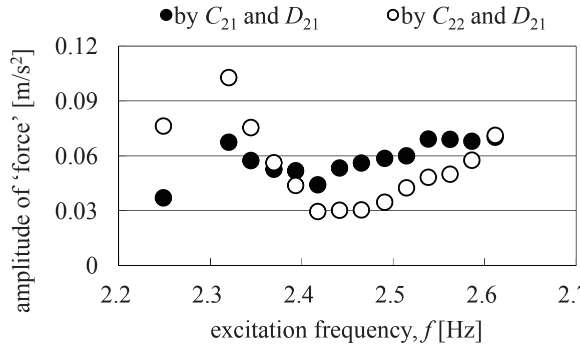


Fig. 6 Amplitudes of the asymmetric (1, 1) mode-synchronous component in sums of nonlinear force terms under each excitation frequency (analysis, $\sigma = 0.50$, $\zeta = 0.02$). The amplitude is represented by $\sqrt{2}$ times the standard deviation of the synchronous component.

◯: incorporated, ×: not incorporated

case name	frequency		incorporated modes				
	increa-sing	decrea-sing	asym(1, 1) (C_{11})	sym(2, 1) (D_{21})	asym(2, 1) (C_{21})	asym(2, 2) (C_{22})	other four modes
8-mode	●	○	◯	◯	◯	◯	◯
4-mode	■	□	◯	◯	◯	◯	×
3-mode-1	▲	△	◯	◯	◯	×	×
3-mode-2	◆	◇	◯	◯	×	◯	×

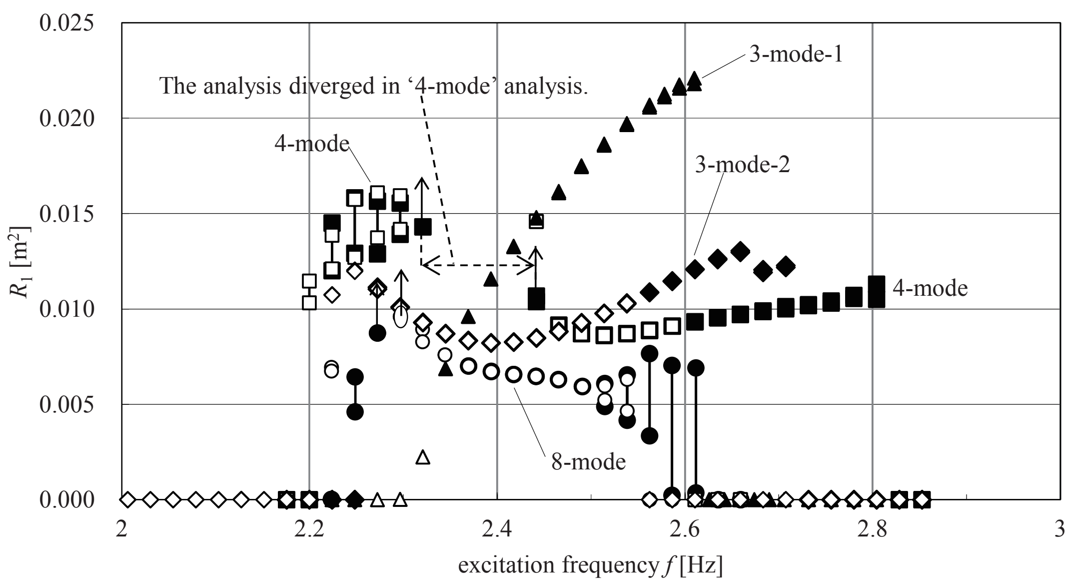


Fig. 7 Amplitude of resonance component of asymmetric (1, 1) mode (R_1) under “peculiar oscillation” (analysis, $\sigma = 0.50$, $\zeta = 0.02$).

and D_{21}) under nonlinear resonance and two asymmetric eigenmodes (C_{21} and C_{22}) that are oscillating at the excitation frequency and are considered to be mediations of the “peculiar oscillation.” The other cases are one case of 3-mode analysis using the two eigenmodes (C_{11} and D_{21}) under nonlinear resonance and one of asymmetric eigenmodes (C_{21}), and one using C_{11}, D_{21} and the other asymmetric eigenmode (C_{22}).

In the remainder of this analysis, R_1 means the amplitude of the resonance component of the asymmetric (1, 1) mode [C_{11}]. R_1 is identified by applying the 4th order Butterworth filter (band-pass filter) to the time-series of C_{11}. The filtering band is the same as that mentioned in Section 5.1. The obtained R_1 values in the different eigenmode combinations are shown in Fig. 7. The vertical line segment in the figure indicates that R_1 modulates within the segment. The actual amplitude by R_1 at the side wall is obtained by $\Phi_{s111}Z'_{s111}(H_1)R_1$ where the coefficient, $\Phi_{s111}Z'_{s111}(H_1)$, is 6.82 [1/m] in the present case.

As shown in Fig. 7, the "peculiar oscillation" occurs (i.e., R_1 develops) not only in the original analysis ('8-mode') but also in the other three cases ('4-mode', '3-mode-1' and '3-mode-2') in which some eigenmodes are omitted. This is in agreement with the fact that two sums of nonlinear terms (the sum consisting of C_{21} and D_{21} and that consisting of C_{22} and D_{21}) act on C_{11} in parallel to cause the "peculiar oscillation".

However, the amplitude and the excitation frequency range of the "peculiar oscillation" are affected by the combination of eigenmodes as shown in Fig. 7. The original analysis ('8-mode') gives amplitudes generally close to the experimental ones (Fig. 2). In contrast, all the other three cases ('4-mode', '3-mode-1' and '3-mode-2') generally give larger amplitudes than that in the original analysis. This implies that nonlinear interferences among eigenmodes restrict the amplitude at the same time as causing the "peculiar oscillation".

It is noticeable that the "peculiar oscillation" occurs in the case of the 3 mode-analysis ('3-mode-1') incorporating the asymmetric (1, 1) mode [C_{11}], the symmetric (2, 1) mode [D_{21}], and the asymmetric (2, 1) mode [C_{21}]. In this 3-mode analysis, the asymmetric (2, 2) mode [C_{22}] under the linear resonance is not incorporated. This indicates that any eigenmode under the linear resonance is not necessary for the "peculiar oscillation" to occur.

6 Amplitude equation analysis for main eigenmodes

To clarify the mechanism of nonlinear coupling in the "peculiar oscillation," analyses of the amplitude equations were performed for the 3 mode-analysis incorporating C_{11}, D_{21}, and C_{21} ('3-mode-1' analysis) and that incorporating C_{11}, D_{21}, and C_{22} ('3-mode-2' analysis) both of which demonstrate the "peculiar oscillation" as shown in Section 5.2.

6.1 Formulation of amplitude equations

From Eqs. (24) and (25), nonlinear modal equations for '3-mode-1' (C_{11}, D_{21} and C_{21}) can be written as:

$$\left.\begin{aligned}
&\Gamma_{1s11}\ddot{C}_{11}+2\zeta_{s11}\omega_{s11}\Gamma_{1s11}\dot{C}_{11}+\Gamma_{2s11}C_{11}+\left\{2/(\lambda_{s1}^2a)\right\}P_{g11}\ddot{x}_g\\
&\quad+\frac{1}{2}(P_{1s121111}\ddot{C}_{11}D_{21}+P_{2s121111}\dot{C}_{11}\dot{D}_{21}+P_{5s211111}\ddot{D}_{21}C_{11}+P_{6s211111}\dot{D}_{21}\dot{C}_{11})\\
&\quad+\frac{1}{2}(P_{1s221111}\ddot{C}_{21}D_{21}+P_{2s221111}\dot{C}_{21}\dot{D}_{21}+P_{5s221111}\ddot{D}_{21}C_{21}+P_{6s221111}\dot{D}_{21}\dot{C}_{21})=0,\\
&\Gamma_{1c21}\ddot{D}_{21}+2\zeta_{c21}\omega_{c21}\Gamma_{1c21}\dot{D}_{21}+\Gamma_{2c21}D_{21}\\
&\quad+\frac{1}{2}(P_{1c112111}\ddot{C}_{11}C_{11}+P_{2c112111}\dot{C}_{11}^2)+\frac{1}{2}(P_{1c222111}\ddot{C}_{21}C_{21}+P_{2c222111}\dot{C}_{21}^2)\\
&\quad+\frac{1}{2}(P_{1c122111}\ddot{C}_{11}C_{21}+P_{2c122111}\dot{C}_{11}\dot{C}_{21}+P_{1c212111}\ddot{C}_{21}C_{11}+P_{2c212111}\dot{C}_{21}\dot{C}_{11})=0,\\
&\Gamma_{1s21}\ddot{C}_{21}+2\zeta_{s21}\omega_{s21}\Gamma_{1s21}\dot{C}_{21}+\Gamma_{2s21}C_{21}+\left\{2/(\lambda_{s1}^2a)\right\}P_{g21}\ddot{x}_g\\
&\quad+\frac{1}{2}(P_{1s122111}\ddot{C}_{11}D_{21}+P_{2s122111}\dot{C}_{11}\dot{D}_{21}+P_{5s212111}\ddot{D}_{21}C_{11}+P_{6s212111}\dot{D}_{21}\dot{C}_{11})\\
&\quad+\frac{1}{2}(P_{1s222111}\ddot{C}_{21}D_{21}+P_{2s222111}\dot{C}_{21}\dot{D}_{21}+P_{5s222111}\ddot{D}_{21}C_{21}+P_{6s222111}\dot{D}_{21}\dot{C}_{21})=0,
\end{aligned}\right\}\qquad(26)$$

and those for '3-mode-2' (C_{11}, D_{21} and C_{22}) can be written as:

$$\left.\begin{aligned}
&\Gamma_{1s11}\ddot{C}_{11}+2\zeta_{s11}\omega_{s11}\Gamma_{1s11}\dot{C}_{11}+\Gamma_{2s11}C_{11}+\left\{2/(\lambda_{s1}^2 a)\right\}P_{g11}\ddot{x}_g\\
&\quad+\frac{1}{2}(P_{1s121111}\ddot{C}_{11}D_{21}+P_{2s121111}\dot{C}_{11}\dot{D}_{21}+P_{5s211111}\ddot{D}_{21}C_{11}+P_{6s211111}\dot{D}_{21}\dot{C}_{11})\\
&\quad-\frac{1}{2}(P_{1s221211}\ddot{C}_{22}D_{21}+P_{2s221211}\dot{C}_{22}\dot{D}_{21}+P_{5s221121}\ddot{D}_{21}C_{22}+P_{6s221121}\dot{D}_{21}\dot{C}_{22})=0,\\
&\Gamma_{1c21}\ddot{D}_{21}+2\zeta_{c21}\omega_{c21}\Gamma_{1c21}\dot{D}_{21}+\Gamma_{2c21}D_{21}+\frac{1}{2}(P_{1c112111}\ddot{C}_{11}C_{11}+P_{2c112111}\dot{C}_{11}^2)\\
&\quad-\frac{1}{2}(P_{3c122121}\ddot{C}_{11}C_{22}+P_{4c122121}\dot{C}_{11}\dot{C}_{22}+P_{3c212211}\ddot{C}_{22}C_{11}+P_{4c212211}\dot{C}_{22}\dot{C}_{11})=0,\\
&\Gamma_{1s22}\ddot{C}_{22}+2\zeta_{s22}\omega_{s22}\Gamma_{1s22}\dot{C}_{22}+\Gamma_{2s22}C_{22}-\left\{2/(\lambda_{s2}^2 a)\right\}P_{g22}\ddot{x}_g\\
&\quad-\frac{1}{2}(P_{3s122112}\ddot{C}_{11}D_{21}+P_{4s122112}\dot{C}_{11}\dot{D}_{21}+P_{7s212112}\ddot{D}_{21}C_{11}+P_{8s212112}\dot{D}_{21}\dot{C}_{11})=0,
\end{aligned}\right\}\tag{27}$$

where, $x_g = X_g \cdot \cos\omega t$. By referring to the frequency components of each eigenmode in Fig. 5, modal variables (C_{11}, D_{21}, and C_{2n}) are assumed to have the following form:

$$\left.\begin{aligned}
&C_{11}=R_1\cos(\omega_1 t+\delta_1)+R_5\cos\left[(\omega_2-\omega_1)t+\delta_5\right]+c_{11c}\cos\omega t+c_{11s}\sin\omega t,\\
&D_{21}=R_2\cos(\omega_2 t+\delta_2)+R_4\cos\left[(\omega+\omega_1)t+\delta_4\right],\\
&C_{2n}=R_3\cos(\omega_1 t+\delta_3)+R_6\cos\left[(\omega_2-\omega_1)t+\delta_6\right]+c_{2nc}\cos\omega t+c_{2ns}\sin\omega t,\\
&\omega_1=\left[\omega_{s11}/(\omega_{s11}+\omega_{c21})\right]\omega,\quad \omega_2=\left[\omega_{c21}/(\omega_{s11}+\omega_{c21})\right]\omega.
\end{aligned}\right\}\tag{28}$$

In Eq. (28), C_{2n}, c_{2nc}, and c_{2ns}, are C_{21}, c_{21c} and c_{21s}, respectively in the '3-mode-1' analysis and those are C_{22}, c_{22c}, and c_{22s} respectively in the '3-mode-2' analysis. ω_1 and ω_2 are the nonlinear oscillation frequencies (approximate value) of the asymmetric (1, 1) mode $[C_{11}]$ and the symmetric (2, 1) mode $[D_{21}]$ respectively, and ω is the excitation frequency, R_{1-6} and δ_{1-6} are the amplitudes and phase angles of the nonlinear components, respectively. R_1 is the amplitude of the component of the frequency close to the natural frequency in the asymmetric (1, 1) mode $[C_{11}]$ and R_2 is that close to the natural frequency in the symmetric (2, 1) $[D_{21}]$ mode. $c_{jnc,s}$ are the amplitudes of ω-components that are subject to both the external disturbance and nonlinear forces. Equation (28) is substituted into Eq. (26) in the case of the '3-mode-1' analysis (C_{11}, D_{21}, and C_{21}), and Eq. (28) is substituted into Eq. (27) in the case of the '3-mode-2' analysis (C_{11}, D_{21}, and C_{22}). Then, maintaining the harmonic balance on each frequency yields the amplitude equation in terms of R_{1-6} δ_{1-6} and $c_{jnc,s}$. For both the '3-mode-1' analysis and '3-mode-2' analysis, the amplitude equation can be written in the following form:

$$\left.\begin{aligned}
&\dot{R}_k=R_k(\Delta_{1R_k}-\gamma_k\Omega_k\Xi_{R_kr}+\varpi_k\Xi_{R_ki})/\Delta_{0R_k}\equiv f_k\quad(k=1-6),\\
&\dot{\delta}_k=(\Delta_{2R_k}+\varpi_k\Xi_{R_kr}+\gamma_k\Omega_k\Xi_{R_ki})/\Delta_{0R_k}\equiv f_{k+6}\quad(k=1-6),\\
&\dot{c}_{11c}=(-\zeta_{s11}\omega_{s11}\Psi_{c_{11}r}+\omega\Psi_{c_{11}i})/\Delta_{0c_{11}}\equiv f_{c_{11}c},\quad \dot{c}_{11s}=(-\omega\Psi_{c_{11}r}-\zeta_{s11}\omega_{s11}\Psi_{c_{11}i})/\Delta_{0c_{11}}\equiv f_{c_{11}s},\\
&\dot{c}_{2nc}=(-\zeta_{s2n}\omega_{s2n}\Psi_{c_{2n}r}+\omega\Psi_{c_{2n}i})/\Delta_{0c_{2n}}\equiv f_{c_{2n}c},\quad \dot{c}_{2ns}=(-\omega\Psi_{c_{2n}r}-\zeta_{s2n}\omega_{s2n}\Psi_{c_{2n}i})/\Delta_{0c_{2n}}\equiv f_{c_{2n}s}.
\end{aligned}\right\}\tag{29}$$

The specific expressions for the symbols in Eq. (29) are given in Appendix 3. In summed and differential harmonic oscillations, only the sum or difference of phase angles has meaning, rather than the individual phase angle of each component. Hence, the simultaneous equations of the R_{1-6} system can be written as follows:

$$\left.\begin{aligned}
&\dot{R}_1=f_1,\ \dot{R}_2=f_2,\ \dot{R}_3=f_3,\ \dot{R}_4=f_4, \dot{R}_5=f_5,\ \dot{R}_6=f_6,\\
&\dot{\psi}_{12^+}=f_7+f_8,\ \dot{\psi}_{23^+}=f_8+f_9,\ \dot{\psi}_{14^-}=f_7-f_{10},\ \dot{\psi}_{12^-5}=f_7-f_8+f_{11},\ \dot{\psi}_{12^-6}=f_7-f_8+f_{12},\\
&\dot{c}_{11c}=f_{c_{11}c},\ \dot{c}_{11s}=f_{c_{11}s},\ \dot{c}_{2nc}=f_{c_{2n}c},\ \dot{c}_{2ns}=f_{c_{2n}s}.
\end{aligned}\right\}\tag{30}$$

Equation (30) is solved by using the Runge-Kutta method (RK4) to obtain a steady value set for R_{1-6}, ψ_{12+}, ψ_{23+}, ψ_{14-}, ψ_{12-5}, ψ_{12-6}, c_{11c}, c_{11s}, c_{2nc}, and c_{2ns}.

6.2 Analytical results and relationship among nonlinear components

The steady values for R_1 of the amplitude equation under each excitation frequency are shown in Fig. 8 along with the original time domain solution. R_1 by the amplitude equation of Eq. (30) is in generally good agreement with the original time domain solution by Eqs. (26) and (27) [i.e., Eqs. (24) and (25)]. It should be noticed that effects of c_{11c} and c_{11s} on R_1 are negligible since the maximum R_1 changes by +6% in the '3-mode-1' analysis or –0.5% in the '3-mode-2' analysis by omitting c_{11c} and c_{11s}. It also should be noticed that there are nonlinear components that restrict R_1 and R_2. For example, R_1 increases by 36% (f = 2.419 Hz) by omitting R_5 in the '3-mode-1' analysis, and R_1 increases by 19% (f = 2.706 Hz) by omitting R_4 in the '3- mode-2' analysis. In order to obtain solutions quantitatively close to the original solutions of Eqs. (26) and (27), it is required to incorporate nonlinear components other than nonlinear resonance components of ω_1 and ω_2.

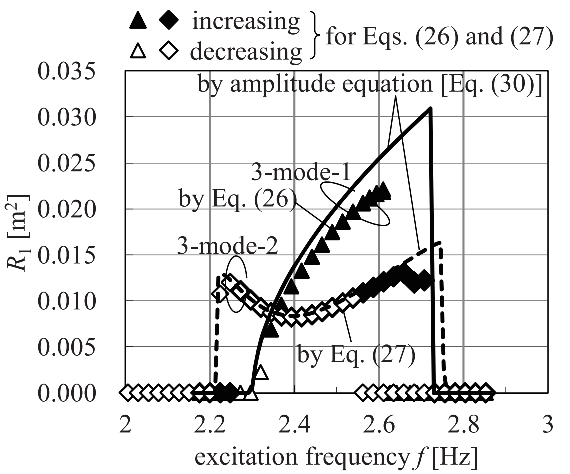


Fig. 8 Solution of R_1 by the amplitude equation ($\sigma = 0.50$, $\zeta = 0.02$).

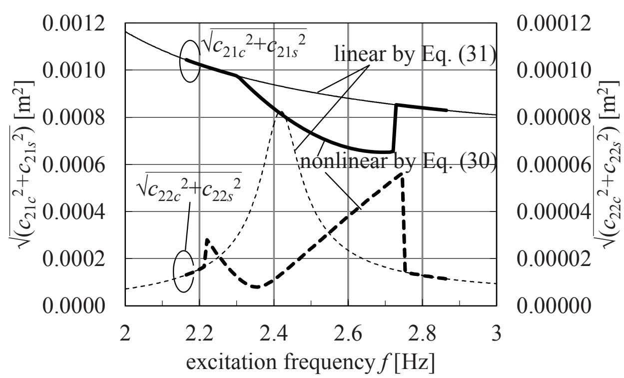


Fig. 9 Excitation frequency (ω) components under "peculiar oscillation" ($\sigma = 0.50$, $\zeta = 0.02$).

The steady amplitudes of ω-components in amplitude equation analyses are shown in Fig. 9, i.e., $\sqrt{(c_{21c}^2 + c_{21s}^2)}$ in the '3-mode-1' analysis and $\sqrt{(c_{22c}^2 + c_{22s}^2)}$ in the '3-mode-2' analysis. Also shown in Fig. 9 are linear solutions obtained by the following equation:

$$\begin{Bmatrix} c_{jnc} \\ c_{jns} \end{Bmatrix} = \frac{-(-1)^n 2\omega^2 P_{gjn} X_g}{\Gamma_{2sjn}\lambda_{sn}^2 a[(1-\chi_{sjn}^2)^2 + 4\zeta_{sjn}^2\chi_{sjn}^2]} \begin{Bmatrix} 1-\chi_{sjn}^2 \\ 2\zeta_{sjn}\chi_{sjn} \end{Bmatrix}, \ \chi_{sjn} = \omega/\omega_{sjn}, \tag{31}$$

where $\Gamma_{2sjn} = \omega_{sjn}^2 \Gamma_{1sjn}$ is used. The amplitude of $\sqrt{(c_{21c}^2 + c_{21s}^2)}$ in the '3-mode-1' analysis is generally close to the linear solution, while $\sqrt{(c_{22c}^2 + c_{22s}^2)}$ in the '3-mode-2' analysis differs from the linear solution. This indicates that the effect of nonlinear terms on ω-components of C_{22} is larger than that of C_{21}. In other words, in eigenmodes with the higher wave number, excitation frequency components tend to be influenced by nonlinear terms.

The following section discusses the relationships among the main three frequency components, i.e., R_1, R_2 (nonlinear resonance components) and $c_{2nc,s}$ (excitation frequency components). For both the '3-mode-1" analysis and '3-mode-2 analysis, Eq. (29) can be rewritten in a form in which the three components are highlighted as follows:

$$\left.\begin{aligned}
\dot{R}_1 &= I_{R_1 1}R_1 + I_{R_1 2}R_2 c_{2nc} + I_{R_1 3}R_2 c_{2ns} + f_{R_1 others},\\
R_1\dot{\delta}_1 &= I_{\delta_1 1}R_1 + I_{\delta_1 2}R_2 c_{2nc} + I_{\delta_1 3}R_2 c_{2ns} + f_{\delta_1 others},\\
\dot{R}_2 &= I_{R_2 1}R_2 + I_{R_2 2}R_1 c_{2nc} + I_{R_2 3}R_1 c_{2ns} + f_{R_2 others},\\
R_2\dot{\delta}_2 &= I_{\delta_2 1}R_2 + I_{\delta_2 2}R_1 c_{2nc} + I_{\delta_2 3}R_1 c_{2ns} + f_{\delta_2 others},\\
\dot{c}_{2nc} &= I_{c_{2nc}1}c_{2nc} + I_{c_{2nc}2}R_1R_2 + I_{c_{2nc}ext}\left\{(-1)^n 2\omega^2 P_{g2n}X_g/(\lambda_{sn}^2 a)\right\} + f_{c_{2nc}others},\\
\dot{c}_{2ns} &= I_{c_{2ns}1}c_{2ns} + I_{c_{2ns}2}R_1R_2 + f_{c_{2ns}others}.
\end{aligned}\right\} \tag{32}$$

Specific expressions for the constants of $I_{R_k 1-3}$, $I_{\delta_k 1-3}$, $I_{c_{2nc,s}1,2,ext}$ and functions of $f_{R_{1,2}others}$, $f_{\delta_{1,2}others}$, $f_{c_{2nc,s}others}$ are omitted here, but can be derived from Eq. (29) with Appendix 3 through a straightforward process.

Based on Eq. (32), the mechanism of the oscillation can be defined as follows. Both nonlinear resonance components R_1 and R_2, form products with $c_{2nc,s}$, i.e., $R_2c_{2nc,s}$ in the equation for $\dot{R}_1$ and $\dot{\delta}_1$, and $R_1c_{2nc,s}$ in the equation for $\dot{R}_2$ and $\dot{\delta}_2$. This indicates that R_1 and R_2 excite each other through the mediation of $c_{2nc,s}$ (excitation frequency component). The schematic of force flows in this relationship is drawn in Fig. 10. This relationship is similar to the equation of the summed and differential harmonic oscillation (combination oscillation) reported by Yamamoto et al. [15]. One different point is that the excitation frequency (ω) component ($c_{2nc,s}$) is affected by nonlinear components of R_1 and R_2 particularly in case of the '3-mode-2' analysis (see Fig. 9). Being based on the force flows, the "peculiar oscillation" is judged to be one of summed and differential harmonic oscillations. The resonance condition in this phenomenon is $f = fs_{11} + fc_{21}$ where f is the excitation frequency and the frequencies on the right-hand side are natural frequencies of eigenmodes under nonlinear resonance. In the present case, these are the natural frequency of the asymmetric (1, 1) mode, fs_{11}, and that of the symmetric (2, 1) mode, fc_{21}. From the view point of engineering, it is useful that the frequency condition for oscillation can be predicted.

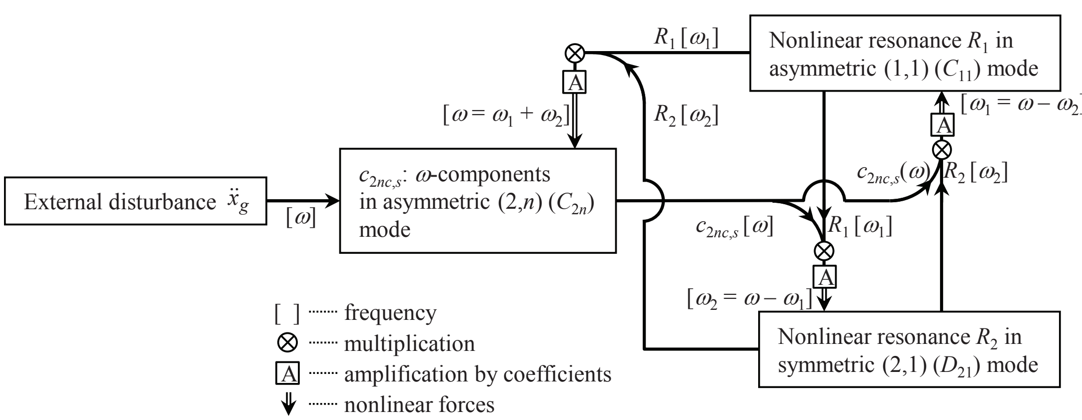


Fig. 10 Schematic of force flows among three modes in the "peculiar oscillation"

7 Conclusions

Nonlinear equations in which linear eigenmodes containing two-interface oscillations couple directly in a nonlinear form were applied to the analysis under experimental conditions. The present analysis demonstrated the nonlinear phenomenon confirmed in the experiment, that is, the interface of two liquids oscillates with an asymmetric first mode shape and at 1/5 to 1/7 of the excitation frequency. The relevant eigenmodes were identified and an amplitude equation analysis was performed on the eigenmodes. It becomes clear that this phenomenon is one of summed and differential harmonic oscillations (combination oscillations) in which the asymmetric first mode with dominant oscillation on the interface of the two liquids and the symmetric first mode with dominant oscillation on the free surface interface are under nonlinear resonance. These eigenmodes excite each other through the mediation of excitation frequency components of two asymmetric modes, i.e., the asymmetric first and second modes with dominant oscillation on the free surface interface.

Appendix 1 Eigenmode analysis

The velocity potentials of modes in each domain may be assumed in the following form:

$$\left.\begin{array}{l}
\phi_1(x,z,t)=\phi_{s1jm}(x,z,t)=\Phi_{s1jm}Z_{s1jm}(z)\sin(\lambda_{sm}x)e^{i\omega_{sjm}t},\quad Z_{s1jm}(z)=\cosh(\lambda_{sm}z)\\
\phi_2(x,z,t)=\phi_{s2jm}(x,z,t)=\Phi_{s2jm}Z_{s2jm}(z)\sin(\lambda_{sm}x)e^{i\omega_{sjm}t},\quad Z_{s2jm}(z)=\cosh(\lambda_{sm}z)+Y_{sjm}\sinh(\lambda_{sm}z)\\
\qquad \ldots\ldots\text{for asymmetric modes,}\\
\phi_1(x,z,t)=\phi_{c1jm}(x,z,t)=\Phi_{c1jm}Z_{c1jm}(z)\cos(\lambda_{cm}x)e^{i\omega_{cjm}t},\quad Z_{c1jm}(z)=\cosh(\lambda_{cm}z)\\
\phi_2(x,z,t)=\phi_{c2jm}(x,z,t)=\Phi_{c2jm}Z_{c2jm}(z)\cos(\lambda_{cm}x)e^{i\omega_{cjm}t},\quad Z_{c2jm}(z)=\cosh(\lambda_{cm}z)+Y_{cjm}\sinh(\lambda_{cm}z)\\
\qquad \ldots\ldots\text{for symmetric modes}
\end{array}\right\}, \tag{33}$$

where ω_{sjm}, ϕ_{sijm}, Φ_{sijm}, and η_{sijm} are the natural angular frequency, the velocity potential of the domain D_i, the amplitude of ϕ_{sijm}, and the wave height of the i-th interface, respectively, in the asymmetric modes. The variables ω_{cjm}, ϕ_{cijm}, Φ_{cijm}, and η_{cijm} have the same meanings in the symmetric modes. m is the wave number order of modes. In Eq. (33), the velocity potential ϕ_1 in the domain D_1 takes a form satisfying the boundary condition at the bottom of the tank, which is a part of Eq. (18). The side wall condition, which consists of Eq. (18) (side wall part), determines the wave number of modes as $\lambda_{sm}=(2m-1)\pi/(2a)$ and $\lambda_{cm}=2m\pi/(2a)$. Linearizing Eq. (16) yields Eq. (34). Substituting Eq. (33) into the first of Eq. (34) yields in Eq. (35).

$$\frac{\partial\eta_1}{\partial t}=\left.\frac{\partial\phi_1}{\partial z}\right|_{z=H_1}=\left.\frac{\partial\phi_2}{\partial z}\right|_{z=H_1},\quad \frac{\partial\eta_2}{\partial t}=\left.\frac{\partial\phi_2}{\partial z}\right|_{z=H_2}. \tag{34}$$

$$\Phi_{s2jm}=\frac{\Phi_{s1jm}\tanh(\lambda_{sm}H_1)}{\tanh(\lambda_{sm}H_1)+Y_{sjm}},\quad \Phi_{c2jm}=\frac{\Phi_{c1jm}\tanh(\lambda_{cm}H_1)}{\tanh(\lambda_{cm}H_1)+Y_{cjm}}. \tag{35}$$

In Eq. (17), after linearizing and eliminating external disturbance terms (x_g), differentiating with respect to time and using Eq. (34) yields:

$$\left.\frac{\partial^2\phi_1}{\partial t^2}\right|_{z=H_1}-\mu\left.\left(\frac{\partial^2\phi_2}{\partial t^2}\right)\right|_{z=H_1}+(1-\mu)g\left.\frac{\partial\phi_1}{\partial z}\right|_{z=H_1}=0, \tag{36}$$

$$\left.\frac{\partial^2\phi_2}{\partial t^2}\right|_{z=H_2}+g\left.\frac{\partial\phi_2}{\partial z}\right|_{z=H_2}=0. \tag{37}$$

Substituting Eq. (33) into Eq. (37) obtains:

$$Y_{sjm}=-\frac{\omega_{sjm}^2\cosh(\lambda_{sm}H_2)-g\lambda_{sm}\sinh(\lambda_{sm}H_2)}{\omega_{sjm}^2\sinh(\lambda_{sm}H_2)-g\lambda_{sm}\cosh(\lambda_{sm}H_2)},\quad Y_{cjm}=-\frac{\omega_{cjm}^2\cosh(\lambda_{cm}H_2)-g\lambda_{cm}\sinh(\lambda_{cm}H_2)}{\omega_{cjm}^2\sinh(\lambda_{cm}H_2)-g\lambda_{cm}\cosh(\lambda_{cm}H_2)}. \tag{38}$$

Substituting Eqs. (33), (35), and (38) into Eq. (36) results in:

$$\left.\begin{aligned}&[1+\mu\tanh(\lambda_{sm}H_1)\tanh(\lambda_{sm}(H_2-H_1))]\,\omega_{sjm}^4-g\lambda_{sm}\,[\tanh(\lambda_{sm}H_1)+\tanh(\lambda_{sm}(H_2-H_1))]\,\omega_{sjm}^2\\&\qquad+(1-\mu)(g\lambda_{sm})^2\tanh(\lambda_{sm}H_1)\tanh(\lambda_{sm}(H_2-H_1))=0,\\&[1+\mu\tanh(\lambda_{cm}H_1)\tanh(\lambda_{cm}(H_2-H_1))]\,\omega_{cjm}^4-g\lambda_{cm}\,[\tanh(\lambda_{cm}H_1)+\tanh(\lambda_{cm}(H_2-H_1))]\,\omega_{cjm}^2\\&\qquad+(1-\mu)(g\lambda_{cm})^2\tanh(\lambda_{cm}H_1)\tanh(\lambda_{cm}(H_2-H_1))=0.\end{aligned}\right\}\tag{39}$$

Solving the quadratic equations in Eq. (39) gives pairs of the second power of the natural angular frequencies, $\omega_{s1m}^2<\omega_{s2m}^2$ and $\omega_{c1m}^2<\omega_{c2m}^2$. Substituting these angular frequencies into Eq. (38) determines Y_{sjm} and Y_{cjm}. Substituting the determined Y_{sjm} and Y_{cjm} into Eq. (35) gives the ratio of Φ_{s2jm} to Φ_{s1jm} and that of Φ_{c2jm} to Φ_{c1jm}, which means that the mode shapes are determined. In the simulation in this paper, mode shapes are normalized by setting $\Phi_{s1jm}=\Phi_{c1jm}=1$.

Appendix 2 Process of the Galarkin method

Substituting Eqs. (21) and (22) into the basic equations of Eqs. (16) and (17) gives the Galerkin method obtaining Eqs. (24) and (25). The process [11] is summarized here. In the substituting process, ϕ_1 and ϕ_2 on F_i are calculated by the following two steps. As an example, first, ϕ_1 is expressed as the Taylor expansion around $z=H_i$, i.e., $\phi_1|_{z=H_i+z_i}=\phi_1|_{z=H_i}+\phi_1'|_{z=H_i}z_i+(1/2)\phi_1''|_{z=H_i}z_i^2$, where $z_i=z-H_i$ and the prime denotes $\partial/\partial z$. Second, the admissible function for η_i, i.e., Eq. (22), substitutes for z_i. After this substituting operation, using the product-to-sum identities for trigonometric functions in integral operations in Eqs. (16) and (17), considering the arbitrary of δA_{jm}, δB_{jm}, δC_{jm}, and δD_{jm}, and retaining the terms of $O(\varepsilon)$ to $O(\varepsilon^3)$ obtains nonlinear equations on A_{jm}, B_{jm}, C_{jm}, and D_{jm}. From the dynamic boundary condition of Eq. (17), the following equations are obtained:

$$G_{1sjn}\dot{A}_{jn}+gG_{2sjn}C_{jn}-(-1)^n\left\{2/(\lambda_{sn}^2a)\right\}P_{gjn}\ddot{x}_g+\Lambda_{1sjn}=0\qquad(j=1,2,n=1,2,\ldots,N),\tag{40}$$

$$G_{1cjn}\dot{B}_{jn}+gG_{2cjn}D_{jn}+\Lambda_{1cjn}=0\qquad(j=1,2,n=1,2,\ldots,N).\tag{41}$$

From the kinematic boundary condition of Eq. (16), the following equations are obtained:

$$-G_{3sjn}A_{jn}+G_{4sjn}\dot{C}_{jn}+\Lambda_{2sjn}=0\qquad(j=1,2,n=1,2,\ldots,N),\tag{42}$$

$$-G_{3cjn}B_{jn}+G_{4cjn}\dot{D}_{jn}+\Lambda_{2cjn}=0\qquad(j=1,2,n=1,2,\ldots,N).\tag{43}$$

The specific expressions for symbols in Eqs. (40) to (43) are given in [11]. Λ_{1sjn}, Λ_{1cjn}, Λ_{2sjn}, and Λ_{2cjn} are nonlinear terms and δ_{nm} in those terms [11] is the Kronecker delta.

By substituting Eqs. (42) and (43) into Eqs. (40) and (41), a nonlinear modal equations of Eqs. (24) and (25) can be obtained that is closed by variables of C_{jn} and D_{jn}. This substitution process is executed in two steps. In the first step, the linear forms of Eqs. (42) and (43) obtained by replacing $\Lambda_{2sjn}$and Λ_{2cjn} with zero $(A_{jn}=(G_{4sjn}/G_{3sjn})\dot{C}_{jn},B_{jn}=(G_{4cjn}/G_{3cjn})\dot{D}_{jn})$, in which the $O(\varepsilon)$ or $O(\varepsilon^{1.5})$ terms are retained, are substituted into $\Lambda_{2sjn}$and Λ_{2cjn} to eliminate A_{jn} and $B_{jn}$from $\Lambda_{2sjn}$and Λ_{2cjn}. In the second step, the resultant equations obtained by the first step are substituted into Eqs. (40) and (41) to eliminateA_{jn} and B_{jn} conclusively.

Appendix 3 Specific expressions for symbols in Equation (29)

Common expressions for the '3-mode-1' (C_{11}, D_{21} and C_{21}) analysis ($n=1$) and the '3-mode-2' (C_{11}, D_{21} and C_{22}) analysis ($n=2$). The relations of $\Gamma_{2sjn}=\omega_{sjn}^2\Gamma_{1sjn}$ and $\Gamma_{2cjn}=\omega_{cjn}^2\Gamma_{1cjn}$ are used in the expressions.

$\gamma_1 = \gamma_5 = \zeta_{s11},\ \gamma_2 = \gamma_4 = \zeta_{c21},\ \gamma_3 = \gamma_6 = \zeta_{s2n},\ \Omega_1 = \Omega_5 = \omega_{s11},\ \Omega_2 = \Omega_4 = \omega_{c21},\ \Omega_3 = \Omega_6 = \omega_{s2n},\ \varpi_1 = \varpi_3 = \omega_1,\ \varpi_2 = \omega_2,$
$\varpi_4 = \omega + \omega_1,\ \varpi_5 = \varpi_6 = \omega_2 - \omega_1,\ \Delta_{0R_i} = 2\Gamma_{1\Delta i}R_i(\varpi_i^2 + \gamma_i^2\Omega_i^2),\ \Delta_{1R_i} = -\gamma_i\Omega_iR_i(\varpi_i^2\Gamma_{1\Delta i} + \Gamma_{2\Delta i}),$
$\Delta_{2R_i} = \varpi_iR_i[-(2\gamma_i^2\Omega_i^2 + \varpi_i^2)\Gamma_{1\Delta i} + \Gamma_{2\Delta i}],\ \Delta_{0c_{11}} = 2\Gamma_{1s11}(\omega^2 + \zeta_{s11}^2\omega_{s11}^2),\ \Delta_{0c_{2n}} = 2\Gamma_{1s2n}(\omega^2 + \zeta_{s2n}^2\omega_{s2n}^2),$
$\Gamma_{1\Delta1} = \Gamma_{1\Delta5} = \Gamma_{1s11},\ \Gamma_{1\Delta2} = \Gamma_{1\Delta4} = \Gamma_{1c21},\ \Gamma_{1\Delta3} = \Gamma_{1\Delta6} = \Gamma_{1s2n},\ \Gamma_{2\Delta1} = \Gamma_{2\Delta5} = \Gamma_{2s11},\ \Gamma_{2\Delta2} = \Gamma_{2\Delta4} = \Gamma_{2c21},\ \Gamma_{2\Delta3} = \Gamma_{2\Delta6} = \Gamma_{2s2n},$
$\Xi_{R_1r} + \sqrt{-1}\Xi_{R_1i} = Q_{1R_2c_{11}}R_2(c_{11c} + \sqrt{-1}c_{11s})e_{12^+} + Q_{1R_4c_{11}}R_4(c_{11c} - \sqrt{-1}c_{11s})e_{14^-} + Q_{1R_2c_{2n}}R_2(c_{2nc} + \sqrt{-1}c_{2ns})e_{12^+}$
$\quad + Q_{1R_4c_{2n}}R_4(c_{2nc} - \sqrt{-1}c_{2ns})e_{14^-} + Q_{1R_2R_5}R_2R_5e_{12^-5} + Q_{1R_2R_6}R_2R_6e_{12^-6},$
$\Xi_{R_2r} + \sqrt{-1}\Xi_{R_2i} = Q_{2R_1c_{11}}R_1(c_{11c} + \sqrt{-1}c_{11s})e_{12^+} + Q_{2R_3c_{2n}}R_3(c_{2nc} + \sqrt{-1}c_{2ns})e_{23^+} + Q_{2R_1c_{2n}}R_1(c_{2nc} + \sqrt{-1}c_{2ns})e_{12^+}$
$\quad + Q_{2R_3c_{11}}R_3(c_{11c} + \sqrt{-1}c_{11s})e_{23^+} + Q_{2R_1R_5}R_1R_5\bar{e}_{12^-5} + Q_{2R_1R_6}R_1R_6\bar{e}_{12^-6} + Q_{2R_3R_5}R_3R_5\bar{e}_{32^-5} + Q_{2R_3R_6}R_3R_6\bar{e}_{32^-6},$
$\Xi_{R_3r} + \sqrt{-1}\Xi_{R_3i} = Q_{3R_2c_{11}}R_2(c_{11c} + \sqrt{-1}c_{11s})e_{23^+} + Q_{3R_4c_{11}}R_4(c_{11c} - \sqrt{-1}c_{11s})e_{34^-} + Q_{3R_2c_{2n}}R_2(c_{2nc} + \sqrt{-1}c_{2ns})e_{23^+}$
$\quad + Q_{3R_4c_{2n}}R_4(c_{2nc} - \sqrt{-1}c_{2ns})e_{34^-} + Q_{3R_2R_5}R_2R_5e_{32^-5} + Q_{3R_2R_6}R_2R_6e_{32^-6},$
$\Xi_{R_4r} + \sqrt{-1}\Xi_{R_4i} = Q_{4R_1c_{11}}R_1(c_{11c} + \sqrt{-1}c_{11s})\bar{e}_{14^-} + Q_{4R_3c_{2n}}R_3(c_{2nc} + \sqrt{-1}c_{2ns})\bar{e}_{34^-} + Q_{4R_1c_{2n}}R_1(c_{2nc} + \sqrt{-1}c_{2ns})\bar{e}_{14^-}$
$\quad + Q_{4R_3c_{11}}R_3(c_{11c} + \sqrt{-1}c_{11s})\bar{e}_{34^-},$
$\Xi_{R_5r} + \sqrt{-1}\Xi_{R_5i} = Q_{5R_1R_2}R_1R_2e_{12^-5} + Q_{5R_3R_2}R_3R_2e_{32^-5},\ \Xi_{R_6r} + \sqrt{-1}\Xi_{R_6i} = Q_{6R_1R_2}R_1R_2e_{12^-6} + Q_{6R_3R_2}R_3R_2e_{32^-6},$
$\Psi_{c_{jn}r} = \Gamma_{1sjn}[(\omega_{sjn}^2 - \omega^2)c_{jnc} + 2\zeta_{sjn}\omega_{sjn}\omega c_{jns}] + (-1)^n[2\omega^2P_{gjn}X_g/(\lambda_{sn}^2a)] + \Xi_{c_{jnc}},$
$\Psi_{c_{jn}i} = \Gamma_{1sjn}[-2\zeta_{sjn}\omega_{sjn}\omega c_{jnc} + (\omega_{sjn}^2 - \omega^2)c_{jns}] + \Xi_{c_{jns}},$
$\Xi_{c_{jnc}} + \sqrt{-1}\Xi_{c_{jns}} = Q_{c_{jn}R_1R_2}R_1R_2\bar{e}_{12^+} + Q_{c_{jn}R_3R_2}R_3R_2\bar{e}_{23^+} + Q_{c_{jn}R_1R_4}R_1R_4e_{14^-} + Q_{c_{jn}R_3R_4}R_3R_4e_{34^-},\ e_{12^+} = \exp(\sqrt{-1}\,\psi_{12^+}),$
$e_{14^-} = \exp(\sqrt{-1}\,\psi_{14^-}),\ e_{23^+} = \exp(\sqrt{-1}\,\psi_{23^+}),\ e_{34^-} = \exp(\sqrt{-1}\,\psi_{34^-}),\ e_{12^-5} = \exp(\sqrt{-1}\,\psi_{12^-5}),\ e_{12^-6} = \exp(\sqrt{-1}\,\psi_{12^-6}),$
$e_{32^-5} = \exp(\sqrt{-1}\,\psi_{32^-5}),\ e_{32^-6} = \exp(\sqrt{-1}\,\psi_{32^-6})$ (The upper bar ($\bar{e}$) denotes the complex conjugate.)
$\psi_{12^+} = \delta_1 + \delta_2,\ \psi_{14^-} = \delta_1 - \delta_4,\ \psi_{12^-5} = \delta_1 - \delta_2 + \delta_5,\ \psi_{12^-6} = \delta_1 - \delta_2 + \delta_6,\ \psi_{23^+} = \delta_2 + \delta_3,$
$\psi_{32^-5} = \delta_3 - \delta_2 + \delta_5 = \psi_{23^+} - \psi_{12^+} + \psi_{12^-5},\ \psi_{32^-6} = \delta_3 - \delta_2 + \delta_6 = \psi_{23^+} - \psi_{12^+} + \psi_{12^-6},\ \psi_{34^-} = \delta_3 - \delta_4 = \psi_{14^-} + \psi_{23^+} - \psi_{12^+}$

Expressions of Q-terms for the '3-mode-1' (C_{11}, D_{21} and C_{21}) analysis

$Q_{1R_2c_{j1}} = (1/4)(-P_{1sj21111}\omega^2 + P_{2sj21111}\omega\omega_2 - P_{5s2j1111}\omega_2^2 + P_{6s2j1111}\omega\omega_2),$
$Q_{1R_4c_{j1}} = (1/4)(-P_{1sj21111}\omega^2 + P_{2sj21111}\omega\omega_{+1} - P_{5s2j1111}\omega_{+1}^2 + P_{6s2j1111}\omega\omega_{+1}),$
$Q_{1R_2R_5} = (1/4)(-P_{1s121111}\omega_{2^-1}^2 + P_{2s121111}\omega_2\omega_{2^-1} - P_{5s211111}\omega_2^2 + P_{6s211111}\omega_2\omega_{2^-1}),$
$Q_{1R_2R_6} = (1/4)(-P_{1s221111}\omega_{2^-1}^2 + P_{2s221111}\omega_2\omega_{2^-1} - P_{5s221111}\omega_2^2 + P_{6s221111}\omega_2\omega_{2^-1}),$
$Q_{2R_1c_{11}} = -(1/4)P_{1c112111}(\omega^2 + \omega_1^2) + (1/2)P_{2c112111}\omega_1\omega,$
$Q_{2R_3c_{21}} = -(1/4)P_{1c222111}(\omega^2 + \omega_1^2) + (1/2)P_{2c222111}\omega_1\omega,$
$Q_{2R_1c_{21}} = (1/4)(-P_{1c122111}\omega_1^2 + P_{2c122111}\omega\omega_1 - P_{1c212111}\omega^2 + P_{2c212111}\omega\omega_1),$
$Q_{2R_3c_{11}} = (1/4)(-P_{1c212111}\omega_1^2 + P_{2c212111}\omega\omega_1 - P_{1c122111}\omega^2 + P_{2c122111}\omega\omega_1),$
$Q_{2R_1R_5} = -(1/4)P_{1c112111}(\omega_1^2 + \omega_{2^-1}^2) - (1/2)P_{2c112111}\omega_1\omega_{2^-1},$
$Q_{2R_1R_6} = -(1/4)(P_{1c122111}\omega_1^2 + P_{2c122111}\omega_1\omega_{2^-1} + P_{1c212111}\omega_{2^-1}^2 + P_{2c212111}\omega_1\omega_{2^-1}),$
$Q_{2R_3R_5} = -(1/4)(P_{1c122111}\omega_{2^-1}^2 + P_{2c122111}\omega_1\omega_{2^-1} + P_{1c212111}\omega_1^2 + P_{2c212111}\omega_1\omega_{2^-1}),$
$Q_{2R_3R_6} = -(1/4)P_{1c222111}(\omega_1^2 + \omega_{2^-1}^2) - (1/2)P_{2c222111}\omega_1\omega_{2^-1},$
$Q_{3R_2c_{j1}} = (1/4)(-P_{1sj22111}\omega^2 + P_{2sj22111}\omega\omega_2 - P_{5s2j2111}\omega_2^2 + P_{6s2j2111}\omega\omega_2),$
$Q_{3R_4c_{j1}} = (1/4)(-P_{1sj22111}\omega^2 + P_{2sj22111}\omega\omega_{+1} - P_{5s2j2111}\omega_{+1}^2 + P_{6s2j2111}\omega\omega_{+1}),$
$Q_{3R_2R_5} = (1/4)(-P_{1s122111}\omega_{2^-1}^2 + P_{2s122111}\omega_2\omega_{2^-1} - P_{5s212111}\omega_2^2 + P_{6s212111}\omega_2\omega_{2^-1}),$
$Q_{3R_2R_6} = (1/4)(-P_{1s222111}\omega_{2^-1}^2 + P_{2s222111}\omega_2\omega_{2^-1} - P_{5s222111}\omega_2^2 + P_{6s222111}\omega_2\omega_{2^-1}),$
$Q_{4R_1c_{11}} = -(1/4)P_{1c112111}(\omega^2 + \omega_1^2) - (1/2)P_{2c112111}\omega_1\omega,$
$Q_{4R_3c_{21}} = -(1/4)P_{1c222111}(\omega^2 + \omega_1^2) - (1/2)P_{2c222111}\omega_1\omega,$
$Q_{4R_1c_{21}} = -(1/4)(P_{1c122111}\omega_1^2 + P_{2c122111}\omega\omega_1 + P_{1c212111}\omega^2 + P_{2c212111}\omega\omega_1),$
$Q_{4R_3c_{11}} = -(1/4)(P_{1c212111}\omega_1^2 + P_{2c212111}\omega\omega_1 + P_{1c122111}\omega^2 + P_{2c122111}\omega\omega_1),$
$Q_{5R_1R_2} = (1/4)(-P_{1s121111}\omega_1^2 + P_{2s121111}\omega_1\omega_2 - P_{5s211111}\omega_2^2 + P_{6s211111}\omega_1\omega_2),$
$Q_{5R_3R_2} = (1/4)(-P_{1s221111}\omega_1^2 + P_{2s221111}\omega_1\omega_2 - P_{5s221111}\omega_2^2 + P_{6s221111}\omega_1\omega_2),$
$Q_{6R_1R_2} = (1/4)(-P_{1s122111}\omega_1^2 + P_{2s122111}\omega_1\omega_2 - P_{5s212111}\omega_2^2 + P_{6s212111}\omega_1\omega_2).$

$Q_{6R_3R_2} = (1/4)(-P_{1s222111}\omega_1^2 + P_{2s222111}\omega_1\omega_2 - P_{5s222111}\omega_2^2 + P_{6s222111}\omega_1\omega_2)$,
$Q_{c_{j1}R_1R_2} = -(1/4)(P_{1s12j111}\omega_1^2 + P_{2s12j111}\omega_1\omega_2 + P_{5s21j111}\omega_2^2 + P_{6s21j111}\omega_1\omega_2)$,
$Q_{c_{j1}R_3R_2} = -(1/4)(P_{1s22j111}\omega_1^2 + P_{2s22j111}\omega_1\omega_2 + P_{5s22j111}\omega_2^2 + P_{6s22j111}\omega_1\omega_2)$,
$Q_{c_{j1}R_1R_4} = (1/4)(-P_{1s12j111}\omega_1^2 + P_{2s12j111}\omega_1\omega_{+1} - P_{5s21j111}\omega_{+1}^2 + P_{6s21j111}\omega_1\omega_{+1})$,
$Q_{c_{j1}R_3R_4} = (1/4)(-P_{1s22j111}\omega_1^2 + P_{2s22j111}\omega_1\omega_{+1} - P_{5s22j111}\omega_{+1}^2 + P_{6s22j111}\omega_1\omega_{+1})$,
$\omega_{+1} = \omega + \omega_1,\ \omega_{2^-1} = \omega_2 - \omega_1$.

Expressions of Q-terms for the '3-mode-2' (C_{11}, D_{21} and C_{22}) analysis

$Q_{1R_2c_{11}} = (1/4)(-P_{1s121111}\omega^2 + P_{2s121111}\omega\omega_2 - P_{5s211111}\omega_2^2 + P_{6s211111}\omega\omega_2)$,
$Q_{1R_2c_{22}} = (1/4)(P_{1s221211}\omega^2 - P_{2s221211}\omega\omega_2 + P_{5s221121}\omega_2^2 - P_{6s221121}\omega\omega_2)$,
$Q_{1R_4c_{11}} = (1/4)(-P_{1s121111}\omega^2 + P_{2s121111}\omega\omega_{+1} - P_{5s211111}\omega_{+1}^2 + P_{6s211111}\omega\omega_{+1})$,
$Q_{1R_4c_{22}} = (1/4)(P_{1s221211}\omega^2 - P_{2s221211}\omega\omega_{+1} + P_{5s221121}\omega_{+1}^2 - P_{6s221121}\omega\omega_{+1})$,
$Q_{1R_2R_5} = (1/4)(-P_{1s121111}\omega_{2^-1}^2 + P_{2s121111}\omega_2\omega_{2^-1} - P_{5s211111}\omega_2^2 + P_{6s211111}\omega_2\omega_{2^-1})$,
$Q_{1R_2R_6} = (1/4)(P_{1s221211}\omega_{2^-1}^2 - P_{2s221211}\omega_2\omega_{2^-1} + P_{5s221121}\omega_2^2 - P_{6s221121}\omega_2\omega_{2^-1})$,
$Q_{2R_1c_{11}} = -(1/4)P_{1c112111}(\omega^2 + \omega_1^2) + (1/2)P_{2c112111}\omega_1\omega$,
$Q_{2R_3c_{22}} = 0,\ Q_{2R_1c_{22}} = (1/4)(P_{3c122121}\omega_1^2 - P_{4c122121}\omega\omega_1 + P_{3c212211}\omega^2 - P_{4c212211}\omega\omega_1)$,
$Q_{2R_3c_{11}} = (1/4)(P_{3c212211}\omega_1^2 - P_{4c212211}\omega\omega_1 + P_{3c122121}\omega^2 - P_{4c122121}\omega\omega_1)$,
$Q_{2R_1R_5} = -(1/4)P_{1c112111}(\omega_1^2 + \omega_{2^-1}^2) - (1/2)P_{2c112111}\omega_1\omega_{2^-1},\ Q_{2R_3R_6} = 0$,
$Q_{2R_1R_6} = (1/4)(P_{3c122121}\omega_1^2 + P_{4c122121}\omega_1\omega_{2^-1} + P_{3c212211}\omega_{2^-1}^2 + P_{4c212211}\omega_1\omega_{2^-1})$,
$Q_{2R_3R_5} = (1/4)(P_{3c122121}\omega_{2^-1}^2 + P_{4c122121}\omega_1\omega_{2^-1} + P_{3c212211}\omega_1^2 + P_{4c212211}\omega_1\omega_{2^-1})$,
$Q_{3R_2c_{11}} = (1/4)(P_{3s122112}\omega^2 - P_{4s122112}\omega\omega_2 + P_{7s212112}\omega_2^2 - P_{8s212112}\omega\omega_2)$,
$Q_{3R_4c_{11}} = (1/4)(P_{3s122112}\omega^2 - P_{4s122112}\omega\omega_{+1} + P_{7s212112}\omega_{+1}^2 - P_{8s212112}\omega\omega_{+1})$,
$Q_{3R_2R_5} = (1/4)(P_{3s122112}\omega_{2^-1}^2 - P_{4s122112}\omega_2\omega_{2^-1} + P_{7s212112}\omega_2^2 - P_{8s212112}\omega_2\omega_{2^-1})$,
$Q_{3R_2c_{22}} = 0,\ Q_{3R_4c_{22}} = 0,\ Q_{3R_2R_6} = 0$,
$Q_{4R_1c_{11}} = -(1/4)P_{1c112111}(\omega^2 + \omega_1^2) - (1/2)P_{2c112111}\omega_1\omega$,
$Q_{4R_3c_{22}} = 0,\ Q_{4R_1c_{22}} = (1/4)(P_{3c122121}\omega_1^2 + P_{4c122121}\omega\omega_1 + P_{3c212211}\omega^2 + P_{4c212211}\omega\omega_1)$,
$Q_{4R_3c_{11}} = (1/4)(P_{3c212211}\omega_1^2 + P_{4c212211}\omega\omega_1 + P_{3c122121}\omega^2 + P_{4c122121}\omega\omega_1)$,
$Q_{5R_1R_2} = (1/4)(-P_{1s121111}\omega_1^2 + P_{2s121111}\omega_1\omega_2 - P_{5s211111}\omega_2^2 + P_{6s211111}\omega_1\omega_2)$,
$Q_{5R_3R_2} = (1/4)(P_{1s221211}\omega_1^2 - P_{2s221211}\omega_1\omega_2 + P_{5s221121}\omega_2^2 - P_{6s221121}\omega_1\omega_2)$,
$Q_{6R_1R_2} = (1/4)(P_{3s122112}\omega_1^2 - P_{4s122112}\omega_1\omega_2 + P_{7s212112}\omega_2^2 - P_{8s212112}\omega_1\omega_2),\ Q_{6R_3R_2} = 0$,
$Q_{c_{11}R_1R_2} = -(1/4)(P_{1s121111}\omega_1^2 + P_{2s121111}\omega_1\omega_2 + P_{5s211111}\omega_2^2 + P_{6s211111}\omega_1\omega_2)$,
$Q_{c_{11}R_3R_2} = (1/4)(P_{1s221211}\omega_1^2 + P_{2s221211}\omega_1\omega_2 + P_{5s221121}\omega_2^2 + P_{6s221121}\omega_1\omega_2)$,
$Q_{c_{11}R_1R_4} = (1/4)(-P_{1s121111}\omega_1^2 + P_{2s121111}\omega_1\omega_{+1} - P_{5s211111}\omega_{+1}^2 + P_{6s211111}\omega_1\omega_{+1})$,
$Q_{c_{11}R_3R_4} = (1/4)(P_{1s221211}\omega_1^2 - P_{2s221211}\omega_1\omega_{+1} + P_{5s221121}\omega_{+1}^2 - P_{6s221121}\omega_1\omega_{+1})$,
$Q_{c_{22}R_1R_2} = (1/4)(P_{3s122112}\omega_1^2 + P_{4s122112}\omega_1\omega_2 + P_{7s212112}\omega_2^2 + P_{8s212112}\omega_1\omega_2)$,
$Q_{c_{22}R_1R_4} = (1/4)(P_{3s122112}\omega_1^2 - P_{4s122112}\omega_1\omega_{+1} + P_{7s212112}\omega_{+1}^2 - P_{8s212112}\omega_1\omega_{+1})$,
$Q_{c22R_3R_2} = 0,\ Q_{c22R_3R_4} = 0,\ \omega_{+1} = \omega + \omega_1,\ \omega_{2^-1} = \omega_2 - \omega_1$.

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